International portfolio choice and numeraire portfolio in real term

--------*Review and redo the work of "International Portfolio Choice and Corporation Finance: A Synthesis"*

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Abstract

The structure of the theory of international finance largely mirrors that of domestic financial theory. The distinction between these two comes from the economic segmentation across nations. Under the international setting where Purchasing Power Parity does not hold and where consumers have diversified risk tolerances, International portfolio model have to deal implicitly with real term. International asset pricing problem may be solved by partial pricing. In my paper, I will give a review to a synthesis: "International portfolio choice and corporation finance" by Michael Alder and Bernard Dumas (A&D). The major part will be devoted to its international portfolio choice and asset-pricing model. I will introduce numeraire portfolio method in my paper. My original part (p12-17) is to derive the numeraire portfolio in real term, examine its property and redo the main model (CCAPM) of Alder and Dumas by using this new approach. Comments and conclusion are givens in the last part of this paper.

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I. Introduction

The structure of the theory of international finance largely mirrors that of domestic financial theory. The distinction between these two comes from the economic segmentation across nations. In international portfolio theory, nations are often defined as zones of common purchasing power units, or more precisely as subsets of investors who use the same price index in deflating their anticipated monetary returns. National groups of investors are delineated by deviations from purchasing power parity (PPP) which cause them to evaluate the return from the same security differently and thus desire to hold generally different portfolios. This heterogeneity in individuals' evaluation of return plays havoc with the standard separation, aggregation, and asset pricing results of portfolio theory.

In this paper, I will give a review to a synthesis: "International portfolio choice and corporation finance" by Michael Alder and Bernard Dumas (Alder and Dumas). The major part will be devoted to its international portfolio choice and asset-pricing model. I will redo the main model (CCAPM) by using numeraire portfolio method in real term. Comments to the methods used in this synthesis will be given in the last part of this paper. The plan is as following.

Section II reviews the first 3 parts of this synthesis: stochastic behavior of PPP deviations, normal distribution of international rates of return and optimal portfolio choice in the world capital market where PPP is violated. A continuous-time CCAPM model, which deals implicitly with real returns, is used and solved by using traditional methods----Hamilton-Jacobi-Bellman equation.

Section III redoes the model by using numeraire portfolio method. The traditional method the paper used is complicated and the economic intuition from it is not straightforward. A numeraire portfolio in real term is constructed. The CCAPM model is solved and explained by using this real-term numeraire portfolio method, which is proved to be simple and powerful.

Section IV reviews the international asset pricing method and the remaining parts of this synthesis. Standard CAPM is extended to apply to international settings with multi-dimensional measurement units by limiting its scope to pricing some assets relative to the others. Welfare problem, capital market imperfection, and corporation finance policy are briefly addressed.

Conclusion and comments of the methods used in this synthesis are given by the last part of the paper.

II. The main assumptions and model

1) Empirical foundations

In this synthesis, the first order of work is to build a realistic foundation of its international framework, which raises two empirical questions: Is PPP violated? Are international rates of return normally distributed?

The importance of PPP in international finance comes from the way in which investors compute the real return from a given security. PPP is a relationship between weighted average price levels of different countries. It serves as a measure of the similarity of, or difference between consumption opportunities in different countries. If PPP held exactly, that is, the price indices were exactly in line with the exchange rate (eP*/P=1), investors from different countries would view the international real returns identically. Their evaluation of the real returns from the same securities will differ to the extent that PPP is violated.

For PPP to hold exactly, sufficient conditions include homothetic preferences; commodity price parity (CPP) with respect to every good included in the index; identical tastes to guarantee that the compositions of different nations' indices will be identical.

Empirical results suggest that PPP is violated both in short and long horizon. Moreover, the actual behavior of PPP deviations cannot be statistically distinguished from a maritingale. Alder and Dumas gave an economic interpretation as following. Costly commodities trading would cause the law of one price (CPP) to be violated instantaneously and would allow the exchange rate to fluctuate within a band around its PPP level. On the other hand, costless information-efficient foreign exchange cause the current spot exchange rate to be the best predictor of the future exchange rate adjusted for interest rates. Furthermore, since interest rates anticipate inflation, it also causes the current real exchange rate to be the best predictor of the future exchange rate adjusted for the anticipated inflation difference between the two countries. This is the PPP hypothesis and it leads to the martingale model.

Empirical results also suggest that within a finite range returns on international investment may be approximately normal. Utility functions are approximately bounded; the expected utility integral computed over such a normal approximation may be approximately mean-variance. Furthermore, empirical results show that the correlation among international returns is fairly small, which suggests that considerable scope exist for risk reduction from international portfolio diversification.

2) International portfolio choice-----CCAPM

The international portfolio choice model presented in Alder and Dumas's synthesis is a traditional Consumption Capital Asset Pricing Model (CCAPM). The difference here is that CCAPM implicitly deals with real returns under the international setting where PPP deviation exists.

In their model, they considered a world of L+1 countries and currencies. Nominal returns are measured in terms of the L+1st currency. Nominal rate of return given in another currency can easily be translated. There are N nominally risky securities, whose nominal price dynamics in terms of the measurement currency are given by Geometric Brownian Motions (GBM):

$$dY_i / Y_i = \mu_i dt + \sigma_i dz_i \qquad i = 1 \quad \cdots N \tag{1}$$

Where

Y_i is the market value of security i in terms of currency L+1;

μ_i is the instantaneous expected nominal rate of return on security i;

σ_i is the instantaneous standard deviation of the nominal rate of return on security i; and,

z_i is a standard brownian motion.

There is one riskless security (the N+1 1^{st}): interesting earning bank deposit or short-term bond denominated in the measurement currency. The instantaneous nominal rate of return paid on this deposit is denoted r.

They assume there are L+1 countries and currencies, thus L+1 national investor types. The price index P^l of an investor of type l follows GBM.

$$dP^l / P^l = \pi^l dt + \sigma^l{}_\pi dz^l{}_\pi \qquad l = 1 \cdots L+1 \tag{2}$$

Where π^l and $\sigma_\pi{}^l$ are the expected value and standard deviation of the instantaneous rate of inflation as seen by investor l. The superscript l will be dropped whenever they consider one representative investor.

The investors' optimal problem is expressed as,

$$Max \qquad E\int_t^T V(c, P, s)ds \tag{3}$$

s.t.

$$dW = [\sum_{i=1}^N w_i(\mu_i - r) + r]Wdt - C \bullet Wdt - (\sum_{i=1}^N w_i \sigma_i)Wdz_i \tag{4}$$

Where C is the nominal rate of consumption expenditures. V(.) is the indirect utility function of homogeneous of degree zero in C and P. ω_I indicates the investor's portfolio choice. Equation (4) express the investor's wealth dynamics. J (W, P, t) denotes the maximum value of (3). Following the Hamilton-Jacobi-Bellman principle, they get,

$$0 \equiv \underbrace{Max}_{C,\underline{w}}[V(C,P,t) + J_t + J_W \left\{\left[\sum_{i=1}^{N} w_i\left(\mu_i - r\right) + r\right]W - C\right\} + J_P P_\pi$$

$$+ \tfrac{1}{2} J_{W,W} \sum_{i=1}^{N} \sum_{k=1}^{N} w_i w_k \sigma_{i,k} W^2 + \tfrac{1}{2} J_{P,P} \sigma^2{}_\pi P^2 + J_{W,P} \sum_{i=1}^{N} w_i \sigma_{i,\pi} WP] \qquad (5)$$

Homogeneous of degree zero of J(.) in P and W implies

$$J_P \equiv -(W/P) J_W \qquad (5-1)$$

$$J_{P,W} \equiv -(1/P) J_w - (W/P) J_{W,W} \qquad (5-2)$$

$$J_{P,P} \equiv -(W/P) J_{W,P} + (W/P^2) J_W$$
$$\equiv 2(W/P^2) J_W + (W/P)^2 J_{W,W} \qquad (5-3)$$

Substituting into (5)

$$0 \equiv \underset{C,\underline{w}}{Max}[V(C,P,t) + J_t + J_W \{[\sum_{i=1}^{N} w_i(\mu_i - r) + r - \pi + \sigma_\pi^2 - \sum_{i=1}^{N} w_i \sigma_{i,\pi}] - C\}$$
$$+ \tfrac{1}{2} J_{W,W} \{\sum_{i=1}^{N} \sum_{k=1}^{N} w_i w_k \sigma_{i,k} - 2\sum_{i=1}^{N} w_i\, \sigma_{i,\pi} + \sigma_\pi^2\} W^2] \qquad (5-4)$$

and applying first order condition rule, they get

$$V_C = J_W \qquad (6)$$

$$0 = J_W(\mu_i - r - \sigma_{i,\pi}) + J_{W,W}(\sum_{k=1}^{N} w_k \sigma_{i,k} - \sigma_{i,\pi}) W \qquad (7)$$

Defining: $\alpha = -J_w / J_{ww} W$ as investor's risk tolerance, so the required nominal yield on security i is,

$$\mu_i = r + (1 - \frac{1}{\alpha})\sigma_{i,\pi} + \frac{1}{\alpha}\sum_{k=1}^{N} w_k \sigma_{i,k} \tag{8}$$

Solving the optimal portfolio directly in vector notation, they get,

$$\underline{w} = \alpha\left(\frac{\Omega^{-1}(\mu - r\underline{1})}{1 - \underline{1}'\Omega^{-1}(\mu - r\underline{1})}\right) + (1-\alpha)\left(\frac{\Omega^{-1}\varpi}{1 - \underline{1}'\Omega^{-1}\varpi}\right) \tag{9}$$

Ω is the N∗N matrix of instantaneous covariance $\sigma_{i,k}$ of the nominal rates of return on various securities. ω is the N∗1 vector of covariance $\sigma_{i,\pi}$ of the N risky securities return with the investor's rate of inflation.

The optimal portfolio they get is a combination of two component portfolios. The first one with weight α is the portfolio of a logarithmic investor. The formula indicates that its composition is independent of the behavior of commodity prices. This is equivalent to a log transformation: ln(C/P)=ln C – ln P and therefore commodity prices separate out in the objective function and have no influence on the decisions. As a result, the logarithmic component is the same for all investors, irrespective of nationality; a logarithmic investor would be nationless. So they pointed out that the composition is independent of the choice of the measurement currency. The second component portfolio with weight 1-α is the investor's hedge portfolio. It is his global minimum variance portfolio in real terms. By Ito's lemma, the random part of real returns is the random part of nominal returns minus the random part of the rate of inflation. $\Omega^{-1}\omega$ is the regression coefficient vector of the investor's rate of inflation on the various securities' returns. It minimizes the variance of the difference, which is the variance of the real portfolio return. This portfolio thus is the one whose nominal rate of return is the most highly correlated with the investor's rate of inflation. It is the best hedge against inflation. Since rate of inflation is involved in the hedge portfolio, hedge portfolio is independent of expected nominal return (μ) and the choice of measurement currency.

They summarized the results in the following formulae and a theorem.

$$\underline{w}^{l} = \alpha^{l}\,\underline{w}_{\log} + (1-\alpha^{l})\,\underline{w}_{h}^{l} \tag{10}$$

$$\underline{w}_{\log} = \left(\frac{\Omega^{-1}(\mu - r\underline{1})}{1-\underline{1}'\Omega^{-1}(\mu - r\underline{1})} \right) \tag{11}$$

$$\underline{w}_{h}^{l} = \left(\frac{\Omega^{-1}\varpi^{l}}{1-\underline{1}'\Omega^{-1}\varpi^{l}} \right) \qquad l = 1,\ldots L+1 \tag{12}$$

Theorem. (Optimal portfolio strategy for the individual investor). Every investor in the world holds a combination of:

- the universal logarithmic portfolio with weight α.
- his personalized hedge portfolio, which constitutes the best protection against inflation as he perceives it, with weight 1-α.

They computed the logarithmic portfolio as well as the hedge portfolios of U.S. and French investors as they would have been over the years 1971-1979, based on ex post monthly rates of return. The logarithmic portfolio exhibits large positive and negative weights. Negative weights imply borrowing or short selling. This is the component portfolio, which takes advantage of expected rate of return differentials. There are exact opposite entries in Canadian and U.S. dollar deposits, because, during this period, U.S. interest rates were above Canadian ones without offsetting exchange rate changes.

The hedge portfolio are even more striking. An investor's hedge portfolio is almost entirely made up of a nominal bank deposit (or Treasury Bill or short-term bond) denominated in his home currency. The reason is that exchange rate fluctuations are much wider than price level (CPI) fluctuations. As a consequence, investors who are very risk averse prefer to fully bear their home inflation risk rather than to bear exchange rate uncertainty or stock price uncertainty.

In a specific circumstance where each investor is assumed to ignore his home-currency inflation and therefore consider rates of return expressed in home currency units as being real returns. The consequence is that the home currency bank deposit or Treasury Bill is seen as riskless in real terms by national investors. They explain this result on the basis of equation (12). In this case the hedge portfolio reduces to the home deposit. Consider the portfolio problem of an investor of country l; since he assumes that his rate of inflation measured in currency l is zero (or nonrandom), the same rate of inflation translated into the measurement currency L+1 and to be introduced into the covariance vector ω^l, reduces to the rate of change of the (L+1/l)th exchange rate; and since the translated rate of return on the currency l Treasury Bill to be introduced into the covariance matrix Ω is also equal, in its random part, to the same exchange rate change, the regression of inflation on securities which underlies formula (12) will produce a unit coefficient on the currency-l Bill and zeros on all other securities. Hence:

Corollary. (Solnik, Sercu)

When home inflation (measured in home currency and using home consumption weights) is zero (or nonrandom), every investor in the world holds a combination of:

- the universal logarithmic portfolio with weight α.
- his home currency Treasury Bill or bank deposit, with $1-\alpha$.

Consider another circumstance, a given investor consumes both domestic goods, which have a stable price, and foreign goods whose translated prices fluctuate like exchange rate. He composes his purchasing power index accordingly. When following (12), this index is regressed on the various securities' returns, including translated return of foreign Treasury Bills. The weights of hedge portfolio obviously reconstruct the consumption mix.

Corollary. (Kouri and de Macedo)

When various countries' local-currency output prices are nonrandom, the weights falling on foreign currency Treasury Bills or bank deposits in an investor's hedge portfolio replicate his consumption mix according to the origin of the goods.

III. Redo the model by numeraire portfolio

1) Introducing numeraire portfolio approach

Numeraire portfolio is a new approach to financial theory in continuous-time. It is much simpler than Hamilton-Jacobi-Bellman method used in Alder and Dumas's paper and the economic intuition behind this process is straightforward. Here I try to use numeraire portfolio method to redo their CCAPM and give explanations.

A numeraire portfolio, noted Ht, is the value process of an admissible self-financing strategy, such that the H-denominated value process of any strategy X is a martingale in the "historical probability". We can derive almost any of the classical results of finance theory such as APT, CCAPM and Derivative Asset Pricing model, using the numeraire portfolio.

For Capital Asset Pricing Model (CAPM), the following equation is satisfied,[1]

$$\mu_s = r + \sigma_{HS}$$

where μ_s is the instantaneous expected rate of return of an asset and σ_{HS} is the instantaneous covariance of the returns of H and S ($\sigma_{HS}dt=cov(dH/H, dS/S)$). For Consumption CAPM, we get,[2]

$$\mu_s - r = [-C\,U''(C)/U'(C)]\sigma_{SC}$$

2) Deriving Numeraire portfolio in real term

The CCAPM in international setting deals implicitly with real return. In order to redo their international portfolio choice model, I need to derive numeraire portfolio in real term and its property first.

I can express the investor's real wealth as

$$W' = W/P$$

By Ito's lemma and equation (2) and (4), The dynamic of investor's real wealth is derived as following,

$$dW' = \frac{1}{P}dW - \frac{W}{P^2}dP + \frac{W}{P^3}(dP)^2 - \frac{1}{P^2}dPdW$$

$$\frac{dW'}{W'} = \frac{dW}{W} - \frac{dP}{P} + (\frac{dP}{P})^2 - \frac{dP}{P}\cdot\frac{dW}{W}$$
$$= [\sum_{i=1}^{N} w_i\mu_i + (1-\sum_{i=1}^{N} w_i)r - C - \pi + \sigma_\pi^2 - \sum_{i=1}^{N} w_i\sigma_{i,\pi}]dt + \sum_{i=1}^{N} w_i\sigma_i dz_i - \sigma_\pi dz_\pi \qquad (1')$$

1&2 See Appendix

The numeraire portfolio in real term is expressed as,

$$H' = \frac{H}{P}$$

By Ito’s lemma, the real return of numeraire portfolio is derived as,

$$\frac{dH'}{H'} = \frac{dH}{H} - \frac{dP}{P} + (\frac{dP}{P})^2 - \frac{dP}{P}\frac{dH}{H}$$
$$= (r + \left\|\lambda^2\right\| - \pi + \sigma_\pi^2 - \sigma_{H,\pi})dt + (\sigma_H dz_H - \sigma_\pi dz_\pi) \qquad (2')$$

Because each investor uses the same deflator P[l] to calculate his real wealth W' and real numeraire portfolio H', W'/H'=W'H'⁻ should be martingale. The proof is as following:

$$\frac{dW'H'^{-}}{W'H'^{-}} = \frac{dH'^{-}}{H'^{-}} + \frac{dW'}{W'} + \frac{dH'^{-}}{H'^{-}} \cdot \frac{dW'}{W'}$$

$$\frac{dW'H'^{-}}{W'H'^{-}} = \frac{dH'^{-}}{H'^{-}} + \frac{dW'}{W'} + \frac{dH'^{-}}{H'^{-}} \cdot \frac{dW'}{W'}$$
$$= [\mu_W - \pi + \sigma_\pi^2 - \sum w_i \sigma_{i,\pi} - r + \pi - \sigma_{H,\pi}]dt - (\sum w_i \sigma_{i,H} + \sigma_\pi^2 - \sum w_i \sigma_{i,\pi} - \sigma_{H,\pi})dt$$
$$+ \sum_{i=1}^{N} w_i \sigma_i dz_i - \sigma_H dz_H$$

So we get,

If W’/H’ follows martingale, the drift term should be zero. That is:

$$\mu_W - r + \sigma_\pi^2 - \sum w_i \sigma_{i,\pi} - \sigma_{H,\pi} - \sum w_i \sigma_{i,H} - \sigma_\pi^2 + \sum w_i \sigma_{i,\pi} + \sigma_{H,\pi} = 0$$
$$\Leftrightarrow \mu_w - r = \sum w_i \sigma_{i,H}$$

We know this is held in nominal term, so W'/H' also follows martingale.

(3) CAPM in real term

For nominal security Y_i ,

$$\mu_i - r = \sigma_{i,H}$$
$$\Rightarrow \mu_i - r - \sigma_{i,\pi} = \sigma_{i,H} - \sigma_{i,\pi} \qquad (3')$$

For the R.H.S.

$$\sigma_{i,H} - \sigma_{i,\pi} = \operatorname{cov}(\frac{dH'}{H'}, \frac{dY_i}{Y_i}) = E[(\sigma_H dz_H - \sigma_\pi dz_\pi)\sigma_i dz_i]$$

The economic intuition here is clear. R.H.S. of (3') is covariance between the nominal return of security i and the real return of the numeraire portfolio. L.H.S. of (3') reflects the facts that investors predict their portfolio choice on real returns. Expected real return of risky security i is

$$\mu_i' = \mu_i - \pi + \sigma_\pi^2 - \sigma_{i,\pi}$$

Expected real return of riskless security Y_o is

$$r' = r - \pi + \sigma_\pi^2 - \sigma_{r,\pi} \qquad and \qquad \sigma_{r,\pi} = 0$$
$$so \qquad \mu_i' - r' = \mu_i - r - \sigma_{i,\pi}$$

So the CAPM in real term is

$$(\mu_i - r - \sigma_{i,\pi})dt = \operatorname{cov}(\frac{dH'}{H'}, \frac{dY_i}{Y_i}) \qquad (4')$$

(4) Solving CCAPM by numeraire portfolio

First, we rewrite investor's objective function and constrain in real term

$$Max \quad E\int_t^T J(W_t', s)ds$$

$$s.t. \quad W_o' = E(\frac{W_T'}{H_T'})$$

By assumption J() is homogeneous of degree zero in W and P, so J(W,P,S)=J(W',S)

Where dynamic of W' and H' is given by (1') and (2'). The optional solution is

$$J'(W_t') = \frac{\lambda}{H_t} \qquad J'(W_o) = \lambda$$

$$\Rightarrow \frac{dH'}{H'} = -\frac{W' J_{W'W'}}{J_{W'}} \cdot \frac{dW'}{W'}$$

We know,

$$(\mu_i - r - \sigma_{i,\pi}) \quad dt = \mathrm{cov}(\frac{dH'}{H'}, \frac{dY_i}{Y_i})$$

$$= \mathrm{cov}(-\frac{W' J_{ww}}{J_W} \frac{dW'}{W'}, \frac{dY_i}{Y_i})$$

$$= E(-\frac{W' J_{W'W'}}{J_{W'}} (\sum_{k=1}^{N} w_k \sigma_k dz_i - \sigma_\pi dz_\pi) \sigma_i dz_i)$$

That is,

$$\mu_i - r - \sigma_{i,\pi} = -\frac{W' J_{W'W'}}{J_{W'}} (\sum_{k=1}^{N} w_k \sigma_{i,k} - \sigma_{i,\pi})$$

$$J_w = J_{W'} \cdot \frac{1}{P} \quad \Rightarrow \quad -\frac{W' J_{W'W'}}{J_{W'}} = -\frac{W J_{WW}}{J_W}$$

Where W is the normal term and W' is the real term, thus we get

$$\mu_i - r - \sigma_{i,\pi} = -\frac{WJ_{WW}}{J_W}(\sum_{i=1}^{k} w_k \sigma_{i,k} - \sigma_{i,\pi})$$

$$\mu_i = r + (1 - \frac{1}{\alpha})\sigma_{i,\pi} + \frac{1}{\alpha}\sum_{i=1}^{k} w_k \sigma_{i,k}$$

so the result is exactly the same as (8). If we have all the knowledge of numeraire portfolio, we need not derive them in details. The process is simple and the intuition is clear.

IV. International asset pricing method and other

1) Partial pricing

The traditional Capital Asset Pricing model tries to answer the question: what return must a security bring relative to another security so that investor are willing to hold both in the proportion in which they are available. In the international context, the heterogeneous perceptions of real returns, due to PPP deviations, prevent us from answering this question. Alder and Dumas's paper tries to give a solution to this problem.

The nominal yield required on various securities by an investor to hold portfolio ω is given by equation (8) derived from above. We can rewrite it by emphasizing identity l of the investor,

$$\mu_i = r + (1 - 1/\alpha^l)\sigma^l_{i,\pi} + (1/\alpha^l)\sum_{k=1}^{N} w^l_k \sigma_{i,k} \qquad i = 1 \cdots N \qquad (8 \quad repeated)$$

It can be rewritten in the following form:

$$\mu_i = r + \sigma^l_{i,\pi} + (1/\alpha^l)\sum_{k=1}^{N+1} w^l_k (\sigma_{i,k} - \alpha^l_{i,\pi}); \qquad i = 1 \cdots N \qquad (13)$$

According to this formula, a security must bring a nominal return in excess of the nominal rate of interest, which is made up of two premia; inflation premium (second term of right-hand side) and risk premium (last term of right-hand side). Formulae (8) and (13) are valid but unable to be used directly, because we cannot observe the individual portfolio holding ω^l_k. The only portfolio, which is directly observable, is the aggregate one.

$$w_k^m = \sum_l W^l w_k^l / \sum_l W^l \qquad (13-1)$$

They therefore transform (8) into the aggregate level.

$$\mu_i = r + (1 - 1/\alpha^m)\frac{\sum_l (1-\alpha^l)W^l \sigma_{i,\pi}^l}{\sum_l (1-\alpha^l)W^l} + (1/\alpha^m)\sum_{k=1}^{N} w_k^m \sigma_{i,k} \qquad i = 1 \cdots N \qquad (14)$$

$$where \qquad \alpha^m = (\sum_l W^l \alpha^l)/\sum_l W^l)$$

The disappointment is that the second term is still unobservable, because we cannot measure each individual's risk tolerance. The method Alder and Dumas used is to take the expected yields on L nominal risky securities as given and use them to solve for the unknown weights. The weights so obtained can then be substituted back to (14) to compute the required yields on the other n=N-L securities. The effect of this procedure is to make the covariance between rates of return and PPP deviations, which reflects the differences among investors, equal to zero.

They specify γ as coefficients of a regression of the first n securities on the last L ones, such that the residuals are independent of PPP deviations.

$$\sigma^{l}_{i,\pi} - \sum\nolimits_{k=n+1}^{n+L} \gamma_{i,k} \sigma^{l}_{k,\pi} = \sigma^{L+1}_{i,\pi} - \sum\nolimits_{k=n+1}^{n+L} \gamma_{i,k} \sigma^{L+1}_{k,\pi}; \qquad i = i, \cdots, n, \quad l = 1, \cdots, L. \tag{15}$$

Then they get the following CAPM:

$$\mu_i - \sum\nolimits_{n+1}^{n+L} \gamma_{i,k} \mu_k = r(1 - \sum\nolimits_{k=n+1}^{n+L} \gamma_{i,k}) + (1 - 1/\alpha^m)(\sigma^{l}_{i,\pi} - \sum\nolimits_{k=n+1}^{n+L} \gamma_{i,k} \sigma^{l}_{k,\pi}) + (1/\alpha^m)[\sum\nolimits_{j=1}^{N} w^m_j (\sigma_{i,j} - \sum\nolimits_{k'=n+1}^{n+L} \gamma_{i,k'} \sigma_{k',j})]; \qquad i = 1, \cdots, n; \quad \forall l; \tag{16}$$

Where the right-hand side has the same value no matter which national rate of inflation is used.

The economic interpretation is to visualize γ as side investments in the last L securities which would accompany negatively each unit investment in security i. These side investments are chosen in such a way that the net return is independent of PPP deviations between L+1 national investor groups. The side investment thus constitutes a hedge of security i against PPP deviations. It is natural to choose the L, non-measurement currency bonds as the hedging vehicles. The result is a CAPM, which prices stocks relative to L currency bonds, that is, which provides the expected returns on stocks only when L+1 nominal interest rate are given. They give the theorem**,** the net nominal required yield on a security hedge against PPP deviations is given by traditional nominal Capital Asset Pricing Model

In practice, the regression can be estimated by instrumental variable techniques, with the PPP deviations playing the role of the instrumental variables. This is largely equivalent to a 2 stage LS procedure. In the first stage, one regress the exchange rates on the PPP deviations; in the second stage, one regress the stock returns on the fitted values from the first stage. The coefficients in the second stage provide estimates of γ vector, by construction the residuals are independent of PPP deviations.

2) Interest Rate Parity and Forward Exchange Market

$$f_i = r_{L+1} - r_i \tag{18}$$

In a world of L+1 nations, Alder and Dumas have succeeded in pricing all assets except L+1 of them, which are considered as the L+1 local currency bank deposit or Treasury Bills. In pricing them, they use Asset pricing model (14), which, discussed as above, is not directly testable with usual data. Here they emphasize the international structure of interest rates implied by this model,

$$r_i + \theta_i = r_{L+1} + (1 - 1/\alpha^m)(\sum_l (1-\alpha^l) W^l \, {}^l_{i,\pi}) / \sum_l (1-\alpha^l) W^l$$
$$+ (1/\alpha^m) \sum_{k=1}^N w_k^m s_{i,k}; \qquad i = 1, \cdots, L \tag{17}$$

Where

r_i is the nominal interest rate on the currency i bank deposit;

θ_i is the expected value of the instantaneous rate of change of the exchange rate of currency I against the measurement currency L+1;

r_{L+1} is the measurement currency interest rate, so far denoted simply r;

$s_{i,\pi}^l$ is the covariance of exchange rate I with national investor l's rate of inflation; and

$s_{i,k}$ is the covariance of exchange rate I with the translated return on asset k, including for k=n+1 to N, the covariance with the exchange rates themselves.

In international capital market, with absence of arbitrage opportunity, forward exchange rate is set by interest rate parity (IRP). The percentage difference between the forward and spot rates is denoted by f_i, IRP implies

The forward premium equals the interest rates differential. They show that at the equilibrium, there is an uncovered IRP relationship.

$$f_i = \theta_i - (1 - 1/\alpha^m)[(\sum_l (1-\alpha^l) W^l s^l_{i,\pi}) / \sum_l (1-\alpha^l) W^l] - (1/\alpha^m) \sum_{k=1}^{N} w_k^m s_{i,k} \quad (19)$$

or

$$\begin{aligned} f_i = \theta_i &- [(\sum_l (1-\alpha^l) W^l s^l_{i,\pi}) / \sum_l (1-\alpha^l) W^l] \\ &- (1/\alpha^m) \sum_{k=1}^{N+1} w_k^m [s_{i,k} - (\sum_l (1-\alpha^l) W^l s^l_{i,\pi}) / \sum_l (1-\alpha^l) W^l] \end{aligned} \quad (20)$$

This implies that the forward rate is a biased predictor of the spot. The difference between the forward and expected spot is made up of two premia. One is inflation premium. It would exist even in the absence of risk aversion ($1/\alpha^m$ =0) but would disappear under nonrandom measurement-currency inflation and PPP. Under risk neutrality, the expected value of the spot rate deflated must be equal to the forward rate also deflated. The covariance between the spot rate and deflator is the source of this premium. The second premium is a risk premium and it is linked to the covariance of the exchange rate with the real return on the world market portfolio.

This equilibrium model of forward exchange rate is useful. It led to many empirical studies on the exchange market efficiency. There are no direct tests of equation (20) because the problem of estimating risk tolerance under diverse consumption preference has not been solved. What appears most frequently in the literature is time-series analysis of the nominal difference between the forward and spot rates, uncorrected for inflation. This difference reflects the nominal returns to forward speculation or , equivalently, the forecast error if one takes the forward rate as a predictor of the spot. The literature asks one question: Is forecast errors serially uncorrelated? Taken together, the hypothesis is that the forecast error follows a random walk. This provides an indirect test of whether the premia in (20) are zero.

Evidences are against the random walk hypothesis. Levich discovered biases, which were of opposite sign depending on the direction of change of the spot rate. These he attributed not to risk premian, but to transaction costs which penalize speculation and keep the forward below the expected spot when the latter is rising and vice-versa. Other studies indicate that the unconditional mean bias was significantly different from zero and forecast errors were serially correlated. They are consistent with the existence of biases or premia, which fluctuate widely and in a serially correlated fashion. There is a need for a general equilibrium theory, which would identify the exogenous determinants of the premia.

3) other parts

The welfare problem, market imperfection, corporate finance policy have been addressed

In the last three parts of this paper. Alder and Dumas took a closer look at the welfare problem

that may be connected with the randomness of exchange rates. There has been much debate as to when and whether exchange risk is nominal and real and whether it matters or not. The issue is complex: it depends on such factors as how money is introduced into each economy and why it is held; the completeness of capital markets; and how the government raises revenues and uses the proceeds of money creation.

All the models in their paper were built and discussed under perfect capital market. However, they point out that market imperfection has to be confronted, especially in the international setting. Alder and Dumas then gave a brief review to the literature on segmentation. Some of the papers in this area aim to compute equilibrium conditions, others discuss the welfare gains from bridging the investment barriers and the possibility for optimal corporate decisions. Empirically, the severalties of the market imperfections have not yet to be measured.

Finally they turned to the questions of corporate policy. It was focused mainly on foreign exchange risk avoidance, i.e., hedging policy. Corporate hedging would be irrelevant in a complete, perfect, unified international capital market. They explore the difficulties of measuring exposure and the hedging decision in the circumstances in which it may matter.

V. Comments and conclusion

In this synthesis, Alder and Dumas discussed the portfolio choice and asset pricing in a unified worldwide capital market with PPP deviation. They gave a thorough review on relevant topics from both empirical and theoretical aspects.

I focused my review on their continuous-time capital asset pricing models, which were presented in their first five sections. The method they used to solve the CCAPM model is Hamilton-Jacobi-Bellman principal, which is complicated and hard to analyze. I introduced a real-term numeraire portfolio and redid their CCAPM model by using this simple method.

Several other comments to their methods are as following:

As they had mentioned in their paper, the functional forms ((1)&(2)) for the dynamics of stock and commodities prices are stationary Ito's process. Following Lucas critic, the stochastic process of asset pricing should be fully endogenized by the dynamic methods from functional analysis and by imposing the assumption of rational expectation. That is the nonstationary Ito process where the parameter μ and σ would be functions of a vector of state variables. The functions $\mu(.)$ and $\sigma(.)$ would in turn have to be endogenized at equilibrium.

The stationary GBM for asset prices is not consistent with the positive risk averse investor (Lucas). It would only be consistent with risk neutral investor behavior where μ is the rate of subjective discount. Under risk aversion, asset prices multiplied by discounted marginal utilities must follow martingales and so discounted asset prices by themselves generally do not, in contradiction with equation (1). By Eular equation, $E\,[U'(C_t) / U'(C_o)] = (1+r)^t / (1+\rho)^t$, if the subjective discount rate ρ is zero, we get $E_o[U'(C_t)] = (1+r)^t\ U'(C_o)$. However, this is not a specific problem from this paper, but a problem of a fundamental assumption of asset pricing theory. Attempts have been made to solve this problem.

Future study should focus on the problem of estimating risk tolerance under diverse consumption preference. This is the foundation for testing L+1 local currency deposit or

Treasury bill pricing model and the equilibrium model of the forward exchange rate, equation (17), (19), and (20).

Appendix

1) Deriving CAPM by using numeraire portfolio

<u>Proof:</u>
Consider any self-financing asset with price S and let $H^{-}\equiv 1/H$
Since $S_t/H_t=S_tH_t^{-}$ is a martingale, its drift is zero, hence:

$$\frac{d(S_tH_t^-)}{S_tH_t^-}=\frac{dH_t^-}{H_t^-}+\frac{dS_t}{S_t}+\frac{dH_t^-}{H_t^-}\frac{dS_t}{S_t} \quad \text{and} \quad \frac{dH_t^-}{H_t^-}=-\frac{dH_t}{H_t}+(\frac{dH_t}{H_t})^2, \quad \text{thus:}$$

$$\frac{d(S_tH_t^-)}{S_tH_t^-}=\frac{dH_t^-}{H_t^-}+\frac{dS_t}{S_t}-\frac{dH_t}{H_t}\frac{dS_t}{S_t}, \quad \text{and since the drift is zero:}$$

(1) $\mu_{H^-}+\mu_s-\sigma_{HS}=0$

2) Deriving CCAPM by using numeraire portfolio

Consider an agent exhibiting a time additive utility function, who takes consumption and financial decisions in complete markets and who solves:

$$(P)\left[\begin{array}{l} \underset{\underline{x}_t}{Max} \quad E[\int_0^T u_t(C_t)dt+g(W_T)] \\ \qquad s.t. \\ \underline{x}_t\in A; W(0)=W_0 \end{array}\right.$$

Where C_t is an instantaneous consumption at t, W_T the terminal wealth, u_t the instantaneous utility and g a bequest function.

This optimization problem (P) is equivalent to the following (P')3:

$$(P')\left[\begin{array}{l} \underset{W_T}{Max} \quad E[\int u_t(C_t)dt+g(W_T)] \\ W_0=E[\int_0^T \frac{C_t}{H_t}dt+\frac{W_T}{H_T}] \end{array}\right.$$

Then, the solution of (P') can be simply derived from the Euler equation and the demand function C_t^* satisfy:

(2) $u_t^{'}(C_t^*) = \frac{\lambda}{H_t}, \quad u_0^{'}(C_0^*) = \lambda \quad \text{and} \quad g^{'}(W_T^*) = \frac{\lambda}{H_T}$

Where λ is the lagrangian parameter.

$$\text{Thus :} \quad \frac{1}{H_t} = \frac{u_t^{'}(C_t^*)}{u_0^{'}(C_0^*)} \quad \text{and} \quad \frac{1}{H_T} = \frac{g^{'}(W_T^*)}{u_0^{'}(C_0^*)}$$

Equation (2) implies:

$$u_t^{''}(C_t^*)dC_t^* = -\frac{\gamma}{H_t^2}dH_t$$

Dividing by $u_t^{'}(C_t^*)$ and using again (2):

(3) $-\frac{C_t^* u_t^{''}(C_t^*)}{u_t^{'}(C_t^*)}\frac{dC_t^*}{C_t^*} = \frac{dH_t}{H_t}$

Substituting (3) into $\mu_s = r + \sigma_{HS,}$ we obtain the standard CCAPM equation:

(4) $\mu_s - r = [-\frac{C^* u^{''}(C^*)}{u^{'}(C^*)}]\sigma_{SC}$

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