SYST 500/CSI 600 Lecture 8- Partial Notes

Ariela Sofer

Systems of Linear Differential Equations

1. Introduction

- We are concerned with the system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n + f_n(t) \end{aligned}$$

where $x_1, x_2, \ldots x_n$ are functions of t. This is a **system of linear differential equations**. Setting

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{1n} \\ \vdots & \vdots \\ a_{n11} & a_{nn} \end{pmatrix}, \quad f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

we can write the system as

$$x'(t) = Ax(t) + f,$$

or $x' = Ax + f$ in short. If $f = 0$ the system is called **homogeneous**, otherwise it is **nonhomogeneous**.

2. Solution by Elimination

- Consider the homogeneous system

$$\begin{aligned} x_1' &= 4x_1 - x_2 \\ x_2' &= -x_1 + 4x_2 \end{aligned}$$

We can eliminate one of the variable from the equation. For example we can use the first equation to set $x_2 = 4x_1 - x_1'$ Plugging this into the second equation yields

$$\underbrace{4x_1' - x_1''}_{x_2'} = -x_1 + \underbrace{16x_1 - 4x_1'}_{4x_2}.$$

Collecting terms we get

$$x_1'' - 8x_1' + 15x_1 = 0 \Rightarrow C(\lambda) = \lambda^2 - 8\lambda + 15 = 0 \Rightarrow \lambda = 5, 3 \Rightarrow x_1 = c_1e^{3t} + c_2e^{5t}$$

Therefore

$$x_2 = 4x_1 - x_1' = 4c_1e^{3t} + 4c_2e^{5t} - 3c_1e^{3t} - 3c_2e^{5t} = c_1e^{3t} - c_2e^{5t}$$

Thus

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Consider now a nonhomogeneous version of the system above:

$$\begin{aligned} x_1' &= \quad 4x_1 - \quad x_2 + t \\ x_2' &= \quad - x_1 + 4x_2 - t \end{aligned}$$

For example we can use the first equation to eliminate x_2: $x_2 = 4x_1 - x_1' + t$ Plugging this into the second equation yields

$$\underbrace{4x_1' - x_1'' + 1}_{x_2'} = -x_1 + \underbrace{16x_1 - 4x_1' + 4t}_{4x_2} - t.$$

Collecting terms we get $x_1'' - 8x_1' + 15x_1 = -3t + 1$ Now the solution to the homogeneous equation for this is as above $x_[1c] = c_1e^{3t} + c_2e^{5t}$. To get a solution to the nonhomogeneous equation we guess $x_{1p} = A + Bt$. we have

$$\begin{aligned} x_{1p} &= A + Bt \\ x_{1p}' &= B \\ x_{1p}'' &= 0. \end{aligned}$$

Setting up the correspondence table we get

	x_{1p}''	$-8x_{1p}'$	$+15x_{1p}$	equals
1	0	$-8B$	$15A$	1
t	0	0	$15B$	-3

$$\Rightarrow \begin{aligned} 15A - 8B = 1 \\ 15B = -3 \end{aligned} \Rightarrow A = -1/25, B = -1/5,$$

and $x_1 = c_1 + c_2e^{2x} - t/5 - 1/25$. Moving to x_2

$$x_2 = 4x_1 - x_1' + t = (4c_1e^{3t} + 4c_2e^{5t} - 4t/5 - 4/25) - 3c_1e^{3t} - 3c_2e^{5t} - 1/5 + t = c_1e^{3t} - c_2e^{5t} + t/5 - 9/25$$

Thus

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1/5 \\ 1/5 \end{pmatrix} + \begin{pmatrix} -1/25 \\ -9/25 \end{pmatrix}$$

- A more systematic representation of the substitution method is obtained by using the notation $Dx = dx/dt$, collecting the resulting coefficients of each of the variables in each equation and then solving. As an example if we consider our homogeneous problem it can be written us

$$\begin{aligned} (x_1' - 4x_1) + x_2 = 0 \\ x_1 + (x_2' - 4x_2) = 0 \end{aligned} \Rightarrow \begin{aligned} (D-4)x_1 + x_2 = 0 \\ x_1 + (D-4)x_2 = 0 \end{aligned}.$$

We can then use Gaussian elimination. To shorten the presentation here, we note that we can eliminate x_2 by multiplying the first row by (D-4) and then subtracting the second row. This gives

$$(D-4)^2x_1 - x_1 = 0 \Rightarrow (D^2 - 8D + 15)x_1 = 0 \Rightarrow x_1'' - 8x_1' + 15x_1 = 0.$$

This yields the same solution as we found earlier.

3. A Direct Approach

Consider the homogeneous system

$$x'(t) = Ax(t).$$

A solution vector on an interval I is a vector

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

satisfying the system.

- Example

$$X_1 = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } X_2 = e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

  are solution vectors for our ear;ier homogeneous example.

- The set of solution vectors $X_1, X_2, \ldots . X_n$ be a is **linearly dependent** (LD) on an interval I if there exist constants $c_1, c_2, \ldots, c_n$ not all zeros such that

$$c_1X_1 + c_2X_2 + \ldots + c_nX_n = 0$$

  for all t in I. If it is not LD it is **linearly independent** (LI).

- Any set of n LI solution vectors of the homogeneous system on an interval I is said to be a **fundamental set of solutions** on the interval.

- If $X_1, X_2, \ldots, X_n$ are a fundamental set of solutions Then the general solution of the system $x' = Ax$ on the interval is

$$c_1X_1 + c_2X_2 + \ldots + c_nX_n.$$

- Our previous example suggest that a fundamental set of solutions may exist of solution vectors of the form $x(t) = e^{\lambda t}q$, where q is some vector. To verify this we plug our suggested solution into the equation. We obtain

$$\lambda e^{\lambda t}q = Ae^{\lambda t}q \;\Rightarrow\; q = Aq \;\Rightarrow\; Aq = \lambda q.$$

  It follows that λ must be an eigenvalue of A with corresponding eigenvalue q. We therefore have the following:

- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues of A (not necessarily distinct). Then if $q_1, q_2, \ldots, q_n$ are n LI eigenvectors corresponding to λ_i then

$$e^{\lambda_1 t}q_1,\; e^{\lambda_2 t}q_2,\; \ldots,\; e^{\lambda_n t}q_n$$

are n LI independent solutions of $x'(t) = Ax(t)$ and the general solution can be written as

$$c_1 e^{\lambda_1 t} q_1 + c_2 e^{\lambda_2 t} q_2 + \ldots + c_n e^{\lambda_n t} q_n.$$

If λ is a repeated eigenvalue of multiplicity k that does not have k independent eigenvectors it is possible to seek K LI solutions of the form

$$e^{\lambda t}(q_1 + t q_2 + \ldots + t^{k-1} q_k).$$

- Example. Our previous homogeneous problem can be written as

$$x'(t) = Ax(t), \quad A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

We find the eigenvalues and eigenvectors of A:

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 = 0 \Rightarrow \lambda = 3, 5$$

For $\lambda = 3$

$$(A - \lambda I|0) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \approx \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 5$

$$(A - \lambda I|0) = \left(\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right) \approx \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution is then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$