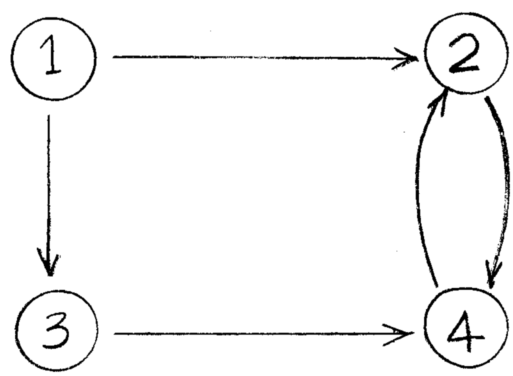
Network Models

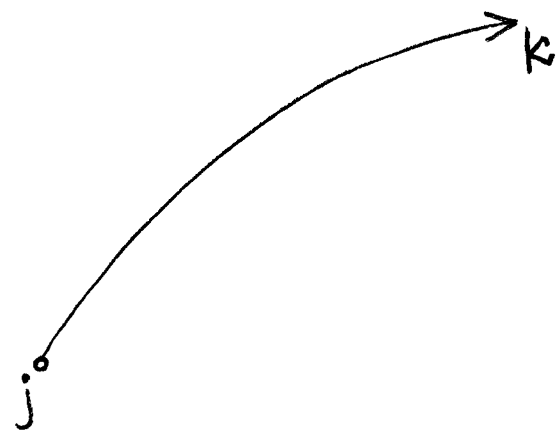
Graph or network is defined by two sets of symbols:

nodes and arcs

points or vertices



Arc consists of an ordered pair of vertices and represents a possible direction of motion



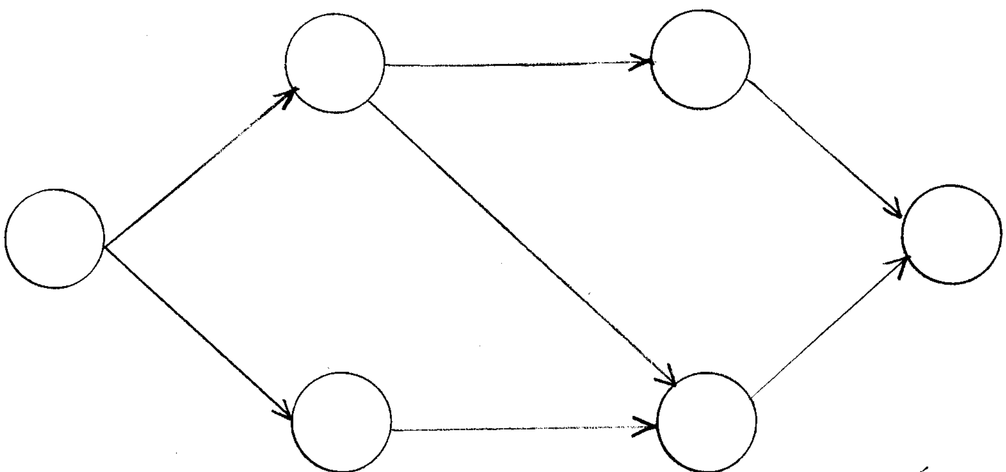
For an arc (j, k) node j is the initial node and node k is the terminal node

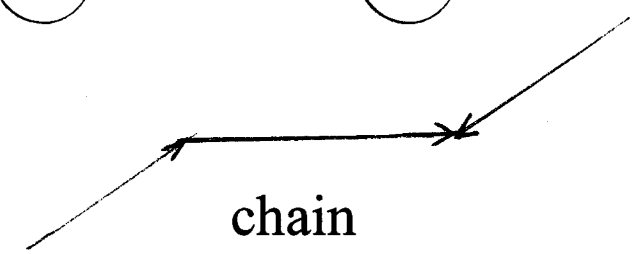
A sequence of arcs such that every arc has exactly one vertex in common with the previous arc is called a <u>chain</u>.

The <u>path</u> is a chain in which the terminal node of each arc is identical to the initial node of the next arc.

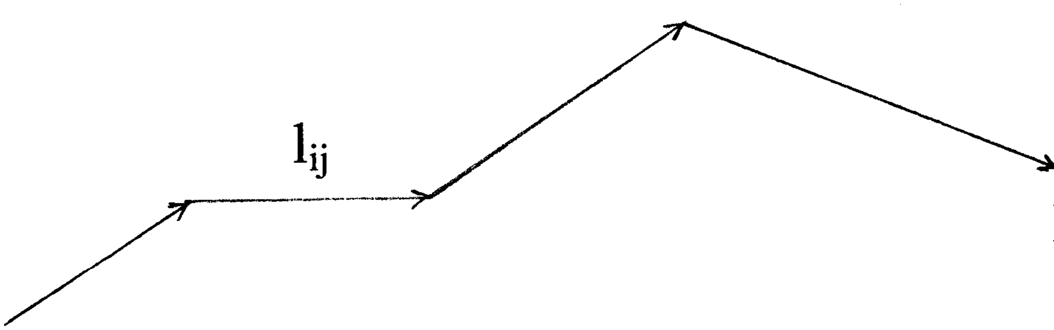
Graph or Network is defined by two sets:
nodes & arcs

G (A, N)





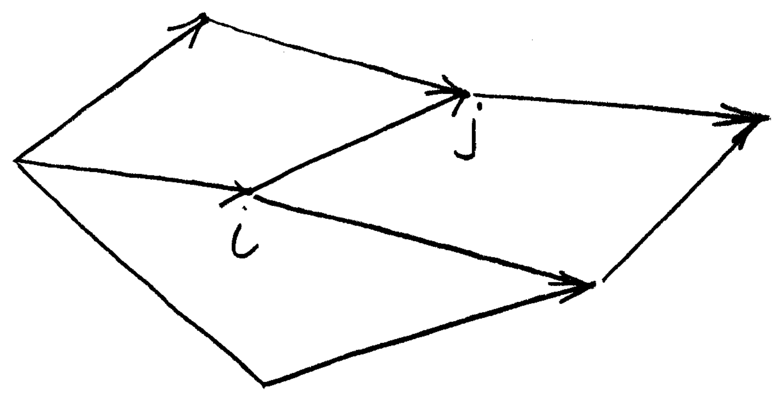
chain



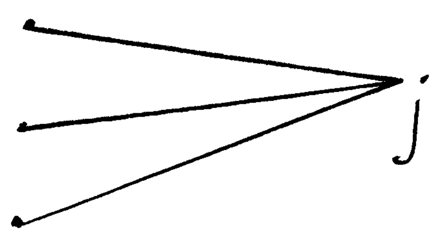
path

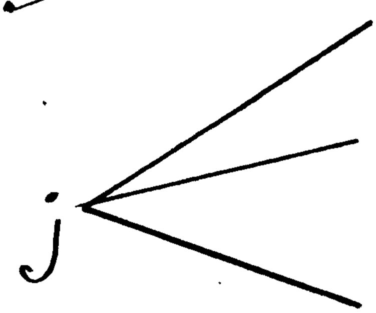
l_{ij} – length between i and j

G (A, V)



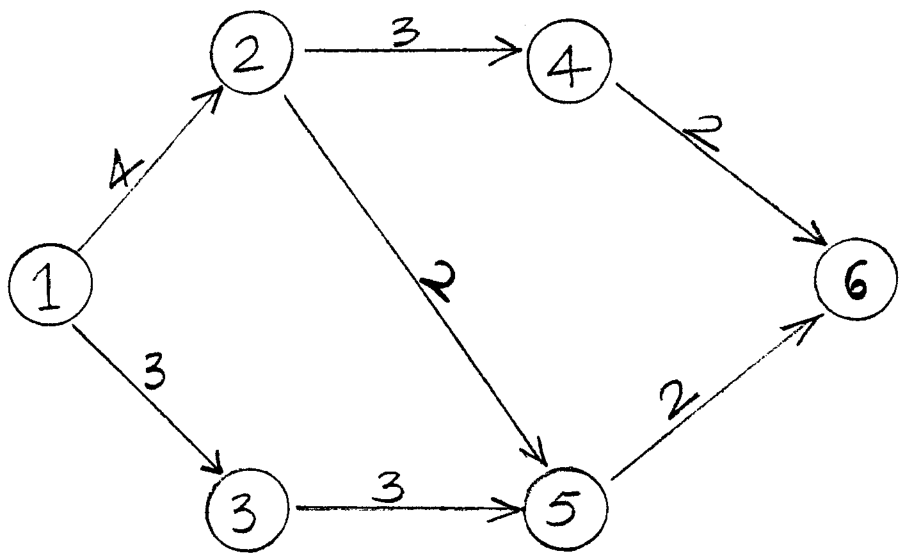
$B(j) = \{i: (i,j) \in G\}$





$A(j) = \{ k: (j,k) \in G\}$

Shortest Path



A path is a chain in which the terminal node of each arc is identical to the initial node of the next arc.

$$x_{ij} = \begin{cases} 1, & if \quad (i,j) \in path \\ 0, & if \quad (i,j) \;\bar{\in}\; path \end{cases}$$

$$z = \sum \sum l_{ij} x_{ij} \rightarrow \min$$

$$\sum_{i \in B(j)} x_{ij} - \sum_{k \in A(j)} x_{jk} = 0$$

$$\sum_{j \in A(0)} x_{oj} = 1, \quad \sum_{i \in B(N)} x_{iN} = 1, \quad 0 \le x_{ij} \le 1$$

$$\sum\sum l_{ij} x_{ij} \to \min$$

$$\sum_{i \in B(j)} x_{ij} - \sum_{k \in A(j)} x_{jk} = 0$$

$$\sum_{j \in A(o)} x_{oj} = 1, \qquad \sum_{i \in B(N)} x_{iN} = 1$$

$$0 \le x_{ij} \le 1$$

Dual

$$\lambda_N - \lambda_0 \to \max$$

$$\lambda_j - \lambda_i \le l_{ij}, \qquad (i,\ j) \in G$$

$$\sum\sum l_{ij} x_{ij}^* = \lambda_N^* - \lambda_o^*$$

λ_j identify with node j, then

$$\lambda_N - \lambda_O \to \max$$

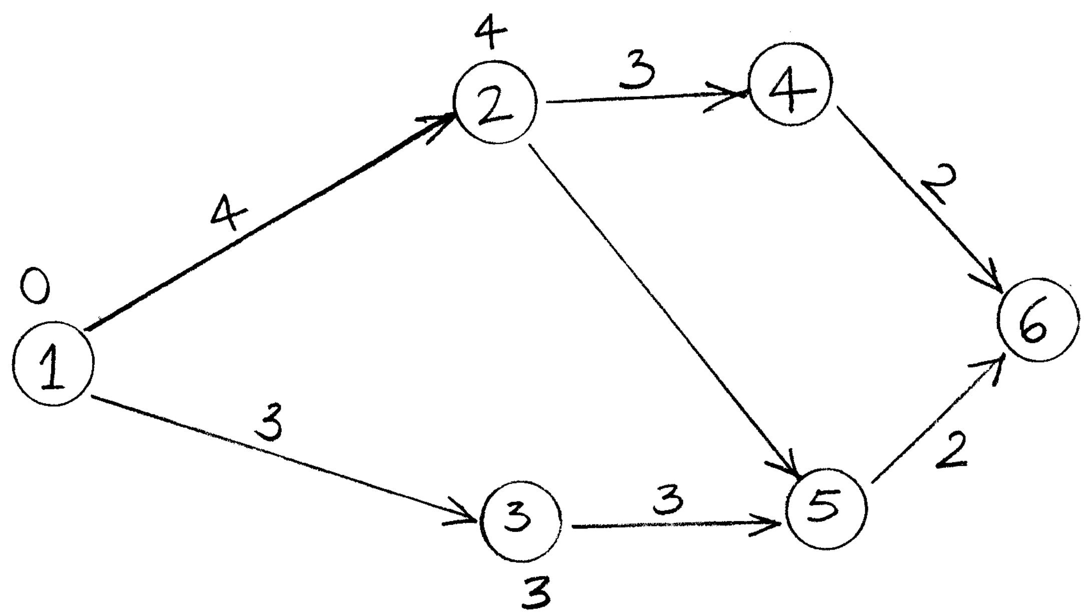
$$\lambda_j - \lambda_i \le l_{ij}$$

$$\sum\sum l_{ij} x^*_{ij} = \lambda^*_N - \lambda^*_O$$

Dijkstra's Algorithm

All labels we consider can be two types: permanent and temporary

The D.A. is a process of labeling and changing temporary for permanent labels.

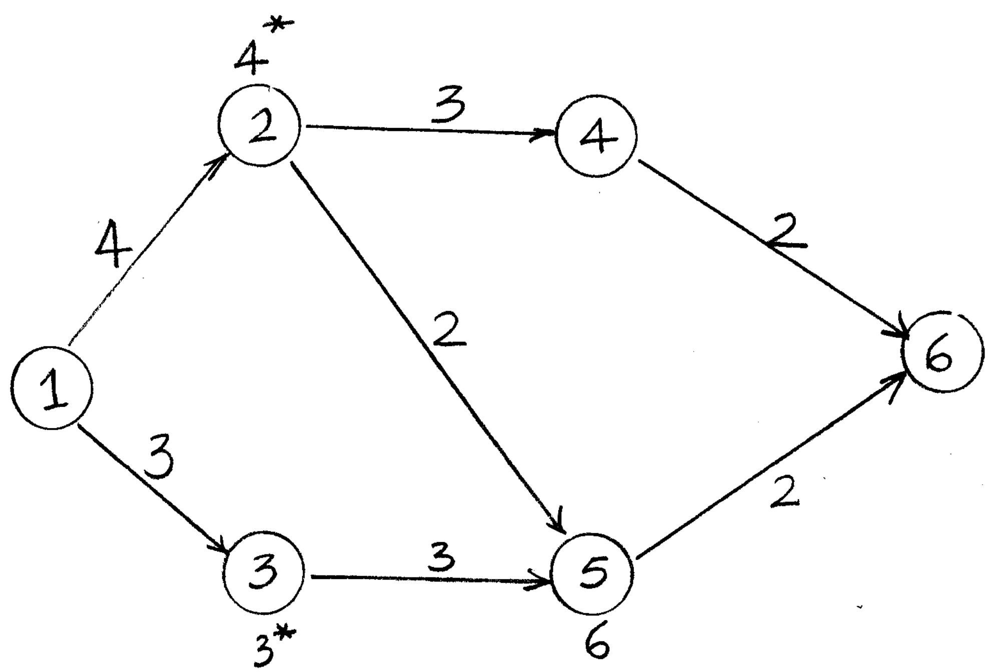


1) The source gets a permanent label 0.

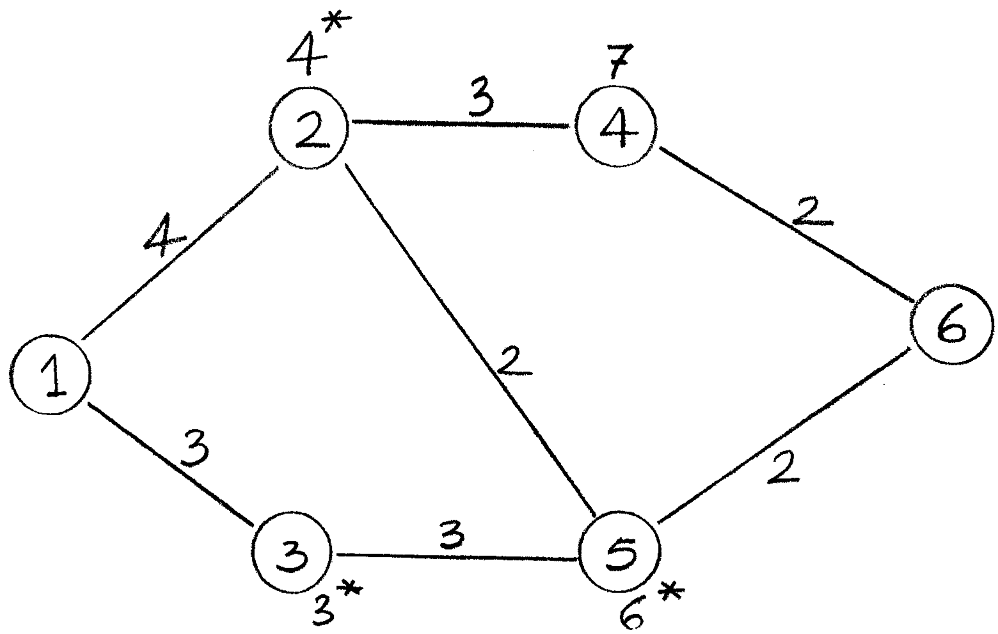
2) We label each node i that is connected to node 1 by a single arc with a "temporary" label equal to the length of the arc (1, i)

3) For the node, which has the smallest temporary label we change the label for a permanent one.
It is clear that 3 is the shortest path from 0 to 3.

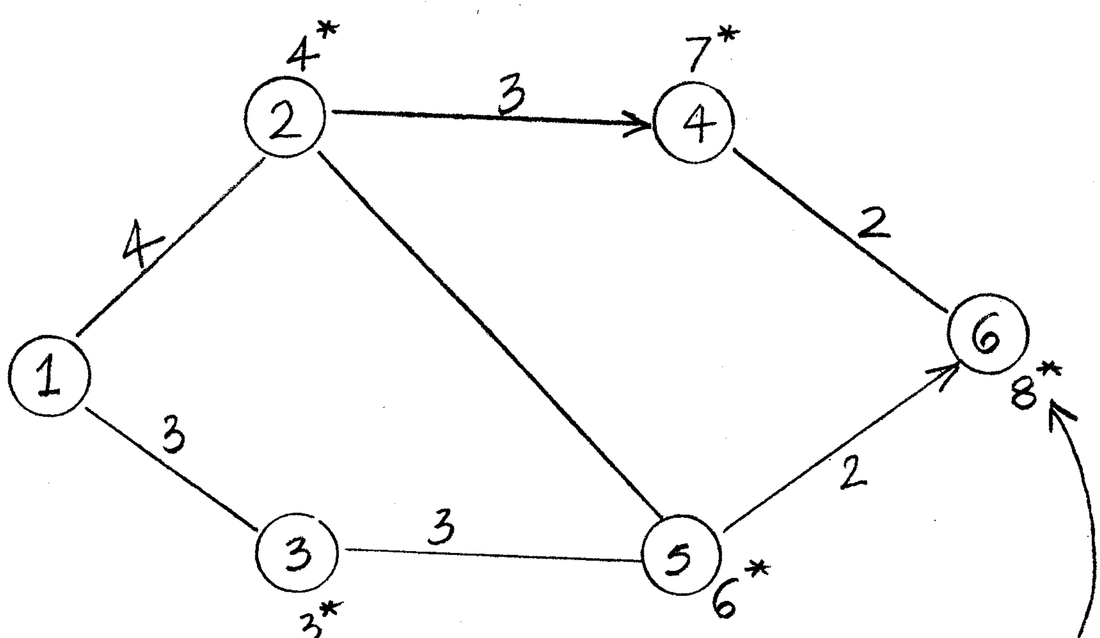
4) Compute temporary label for all nodes connected with 3.



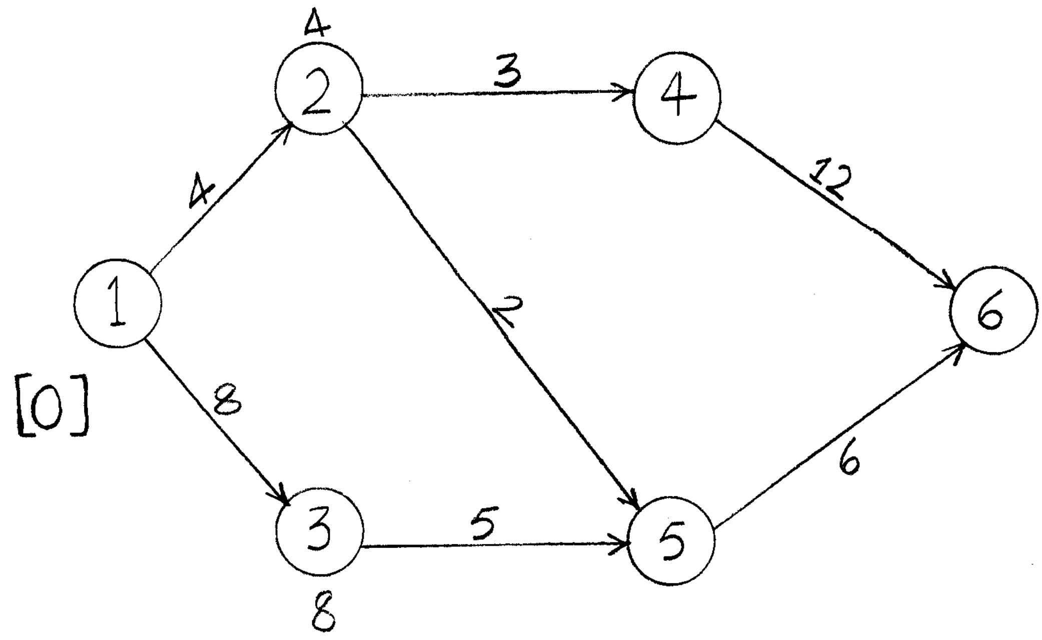
Consider the node with smallest label and change it for permanent.

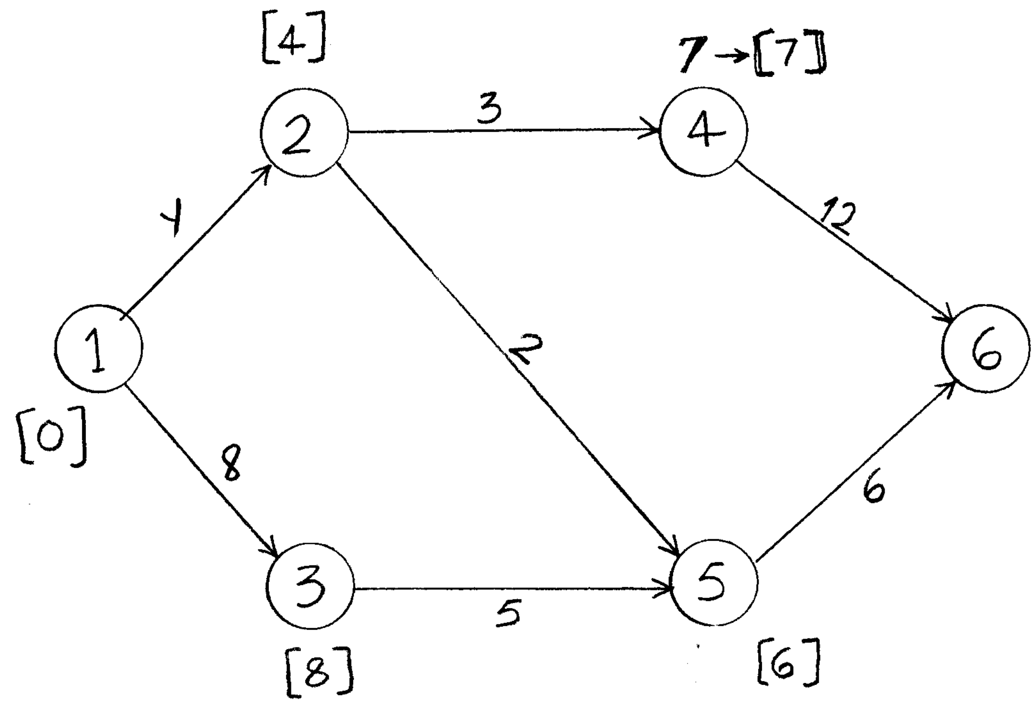


min { 4+2, 3+3 } = 6



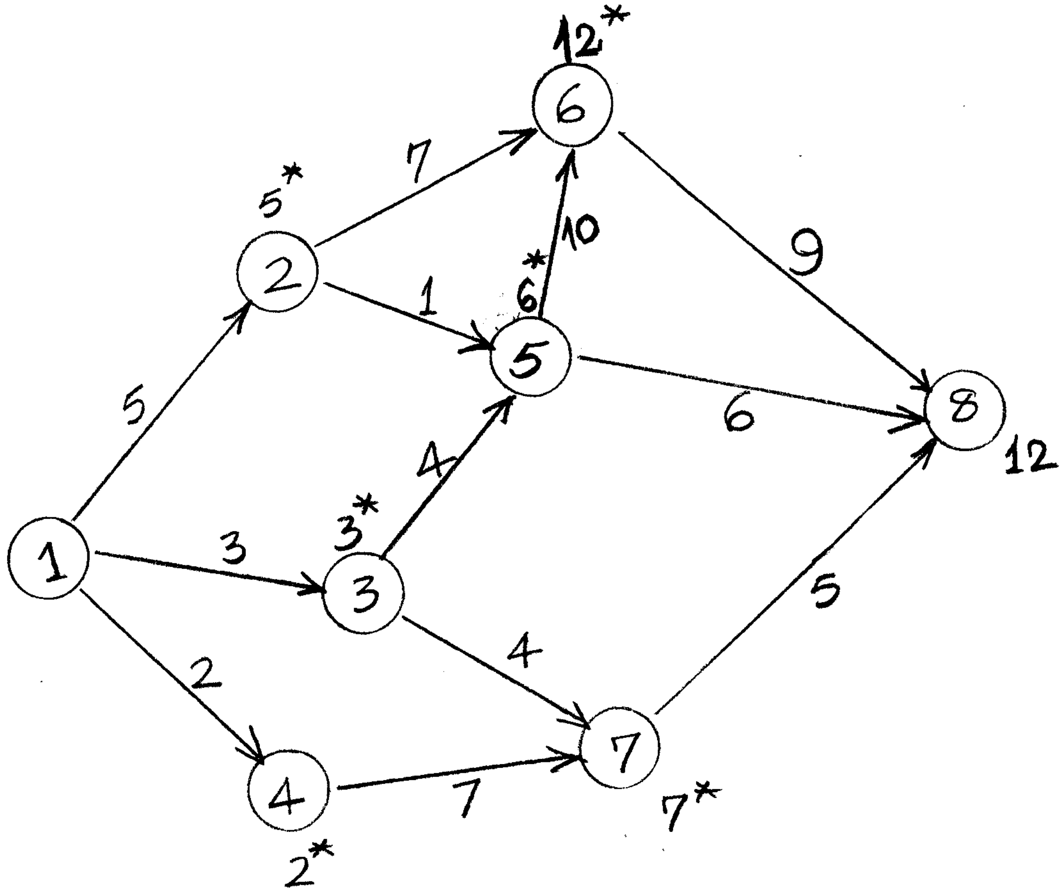
min { 7+2, 6+2 } = 8





min { 8+5, 4+2 } = 6

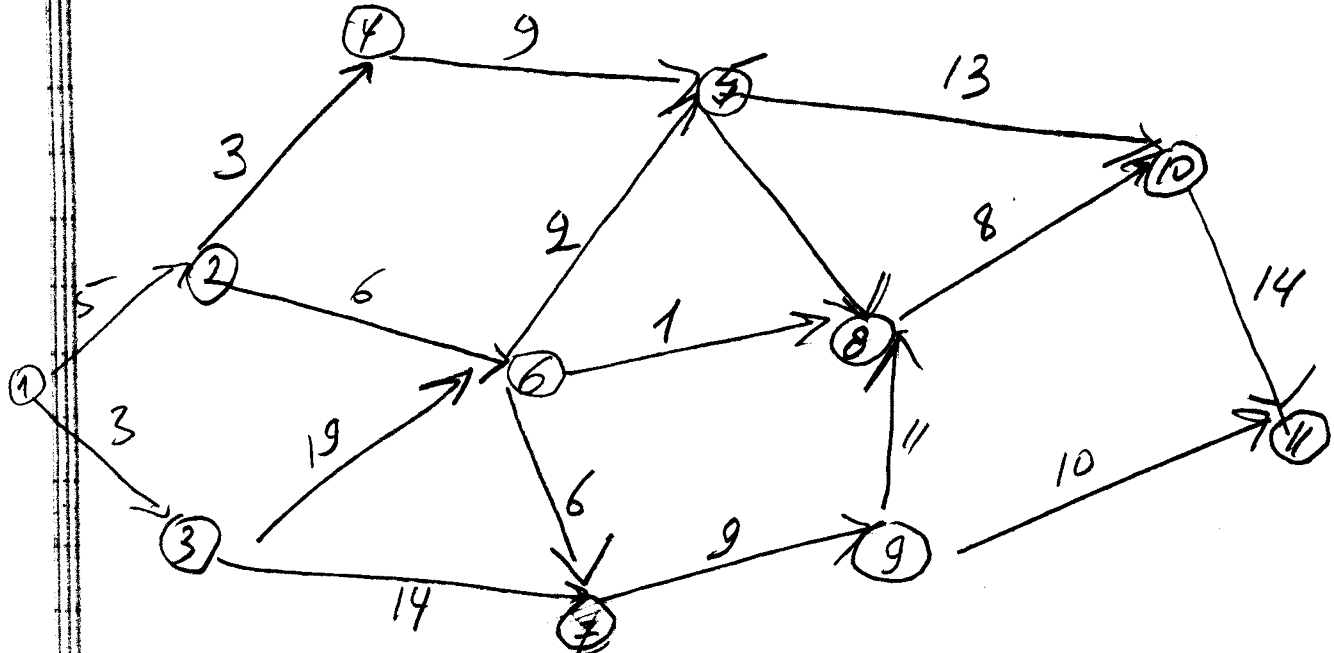
min { 6+6, 7+12 } =12

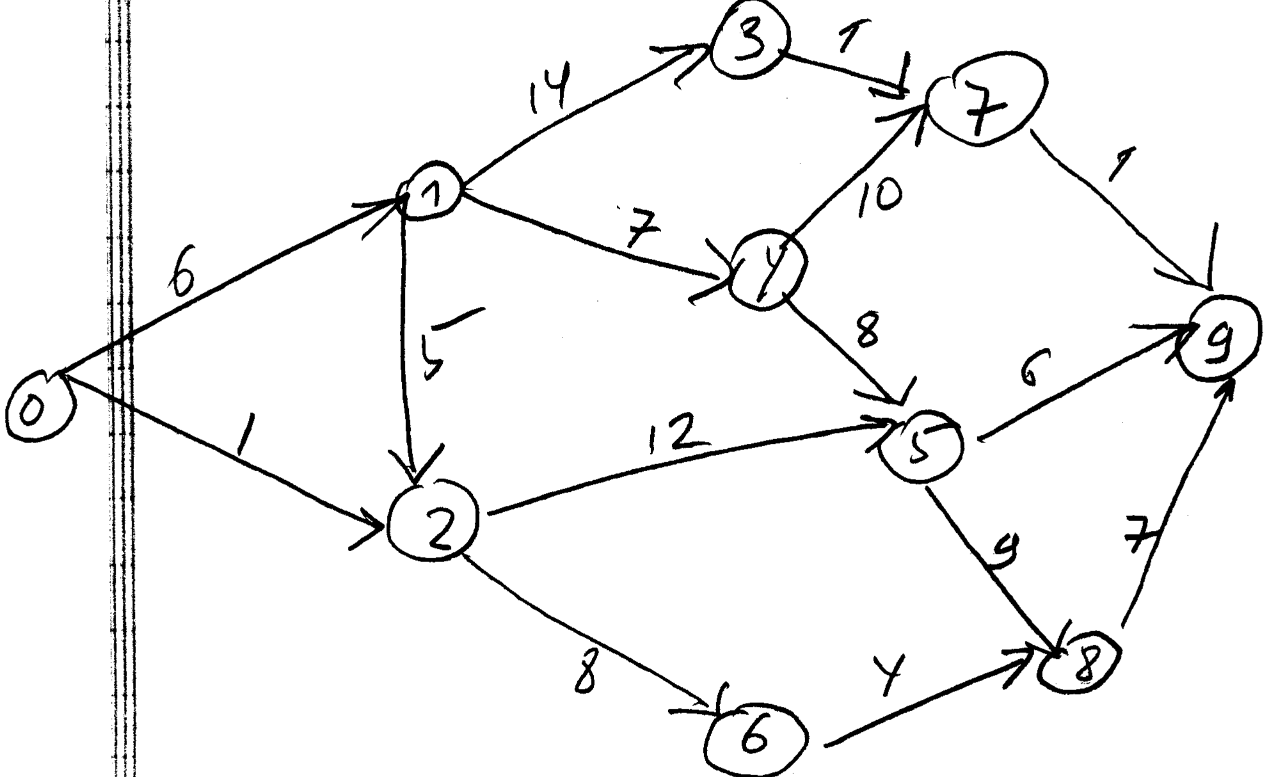


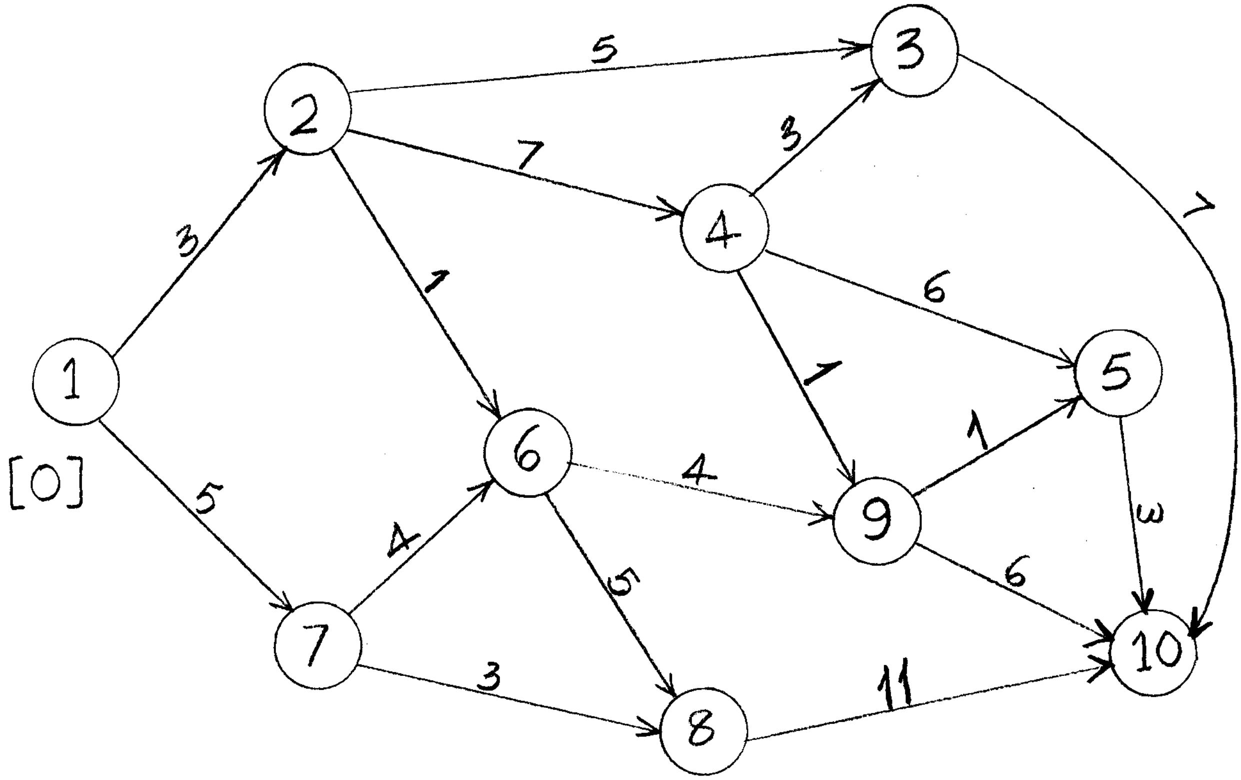
$$\min \{ 5+1, 3+4 \} = 6$$
$$\min \{ 2+7, 3+4 \} = 7$$

$$\min \{5+7,\ 6+10\} = 12$$

$$\min \{ 6+10, 5+7 \} = 12 \Rightarrow 7^*$$
$$\min \{ 12+9, 6+6, 7+5 \} = 12 \Rightarrow 12^*$$


1
2
3
4
5
6
7
8
9
10
11
5
3
3
9
6
19
14
2
1
6
9
13
8
11
10
14


0
1
2
3
4
5
6
7
8
9
6
1
14
7
5
1
10
8
12
8
1
6
9
7
4



(1)

2 → 3 → [3]

7 → 5 → [5]

(2)

3 → 8

4 → 10 → [10]

6 → 4 → [4]

8 → 8

(3)

9 → 11 → [8]

3 → [8]

8 → [8]

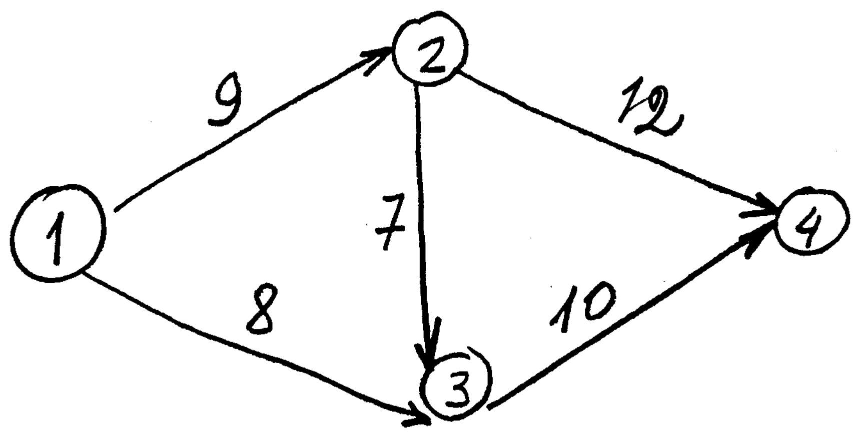
(4)

5 → min{10+6, 11+1} = [12]

(5)

min{8+11, 8+6, 12+3, 8+7} = 15

Max Flow Problems



$\mu(1, 2, 3, 4) = 7$

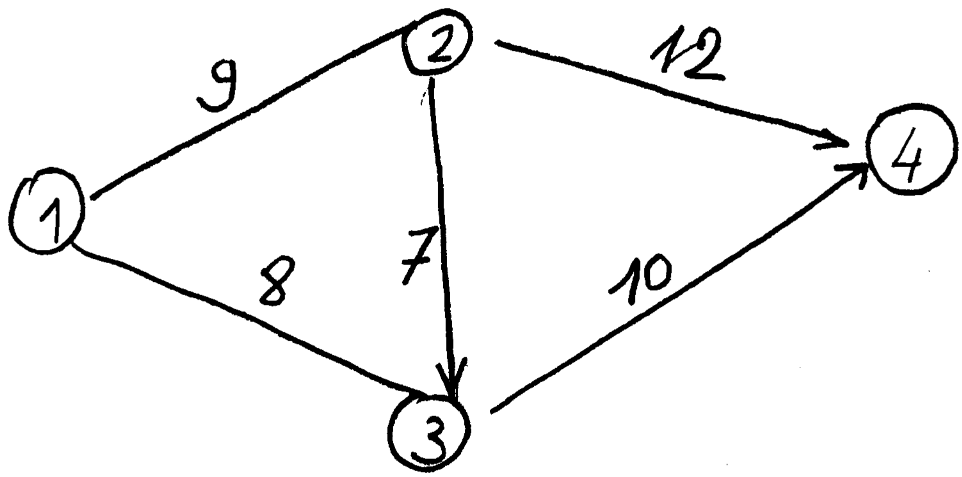
$\mu(1, 2, 4) = 2$

$\mu(1, 3, 4) = 3$

12

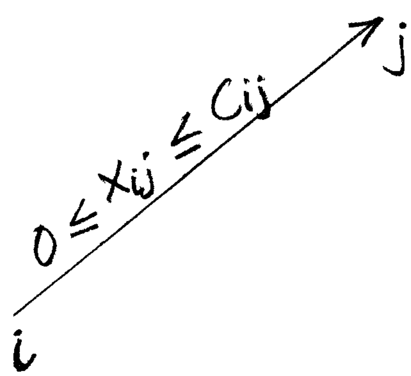
is 12 max flow ?

No



$\mu(1, 2, 4) = 9$

$\mu(1, 3, 4) = 8$



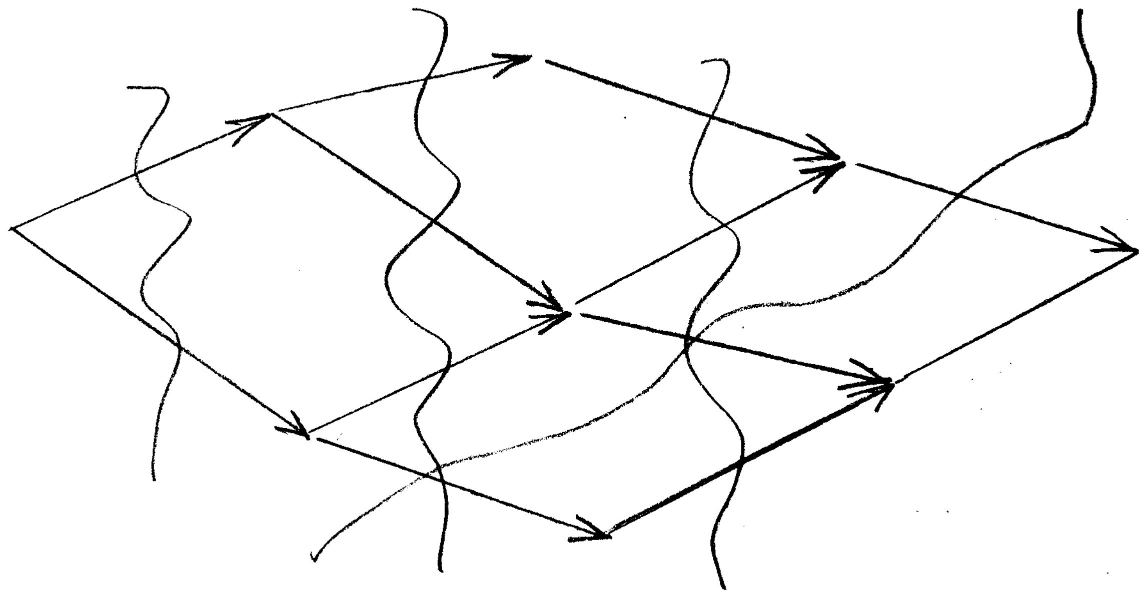
$$0 \le x_{ij} \le c_{ij} \quad - \quad \textit{flow on the arc } (i, j)$$

$$z = \sum_{k \in A(o)} x_{ok} = \sum_{j \in B(n)} x_{jn}$$

$$\sum_{i \in B(j)} x_{ij} - \sum_{k \in A(j)} x_{jk} = 0$$

$$0 \le x_{ij} \le c_{ij}$$

cut



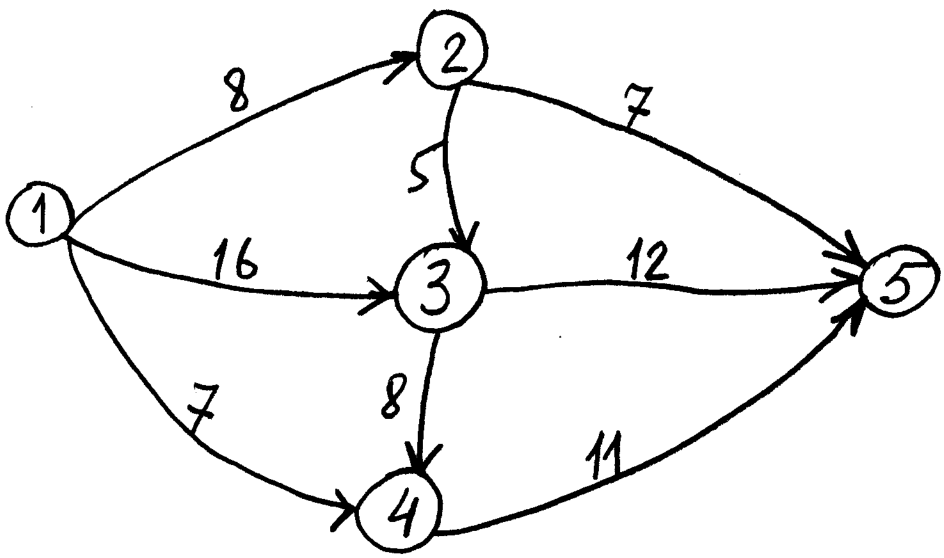
$U, V:$ $\qquad U \cup V = N$

$\qquad\qquad U \cap V = \phi$

$P_o \in U, \quad P_n \in V$

$$c(U, V) = \sum_{i \in U,\ j \in V} c_{ij},$$

$X = x_{ij}$, $\qquad f(X) \le c(U, V)$



$\mu(1, 2, 3, 5) = 5$

$\mu(1, 3, 5) = 7$

$\mu(1, 2, 5) = 3$

$\mu(1, 3, 4, 5) = 4$

$\mu(1, 4, 5) = 7$

$\mu(1, 3, 2, 5) = 4$

$W^* = 30$

$u = \{1, 2, 3, 4\}, \quad v = \{5\}$

$\mu(1, 2, 3, 5) = 5$

$\mu(1, 3, 5) = 7$

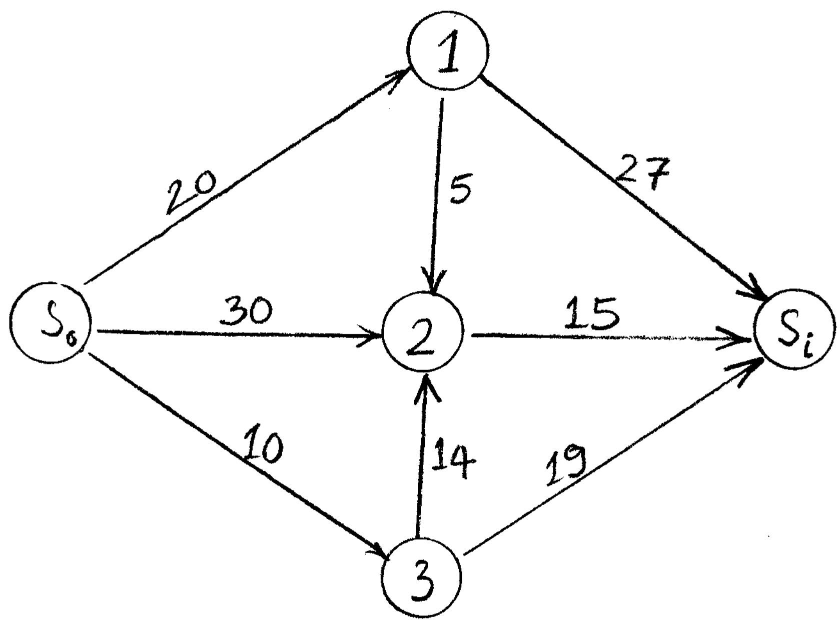
$\mu(1, 2, 5) = 3$

$\mu(1, 3, 4, 5) = 8$

$\mu(1, 4, 5) = 3$

$\mu(1, 4, 3, 2, 5) = 4$

30

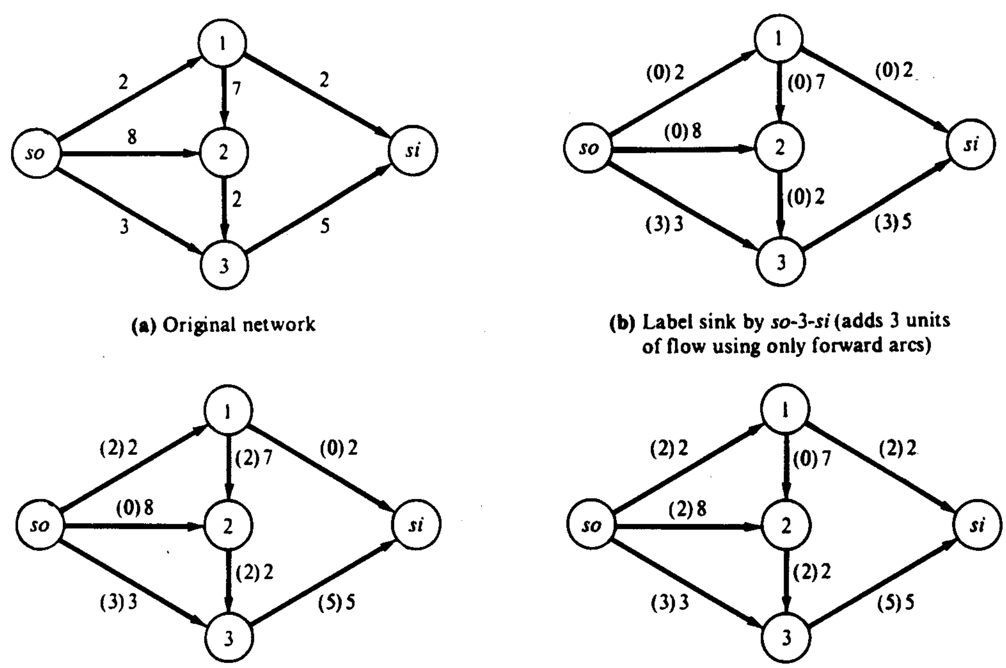


1) General Transportation Problem
   Hungarian Method

2) Assignment Problem

3) Many sources and many sinks

FIGURE 17 Example of Ford–Fulkerson Method



(a) Original network

(b) Label sink by *so*-3-*si* (adds 3 units of flow using only forward arcs)

(c) Label sink by *so*-1-2-3-*si* (adds 2 units of flow using only forward arcs)

(d) Label sink by *so*-2-1-*si* (adds 2 units of flow by using backward arc (1, 2); maximum flow of 7 has been obtained

Problems

Group A

1–3 Figures 18–20 show the networks for Problems 1–3. Find the maximum flow from source to sink in each network. Find a cut in the network whose capacity equals the maximum flow in the network. Also, set up an LP that could be used to determine the maximum flow in the network.

4–5 For the networks in Figures 21 and 22, find the maximum flow from source to sink. Also find a cut whose capacity equals the maximum flow in the network.

6 Seven types of packages are to be delivered by five trucks. There are three packages of each type, and the capacities of the five trucks are 6, 4, 5, 4, and 3 packages, respectively. Set up a maximum flow problem that can be

FIGURE 18 Network for Problem 1

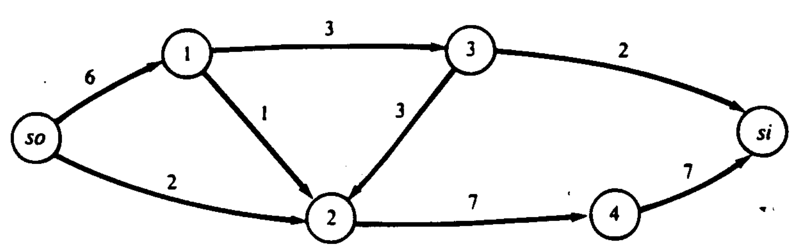


FIGURE 19 Network for Problem 2

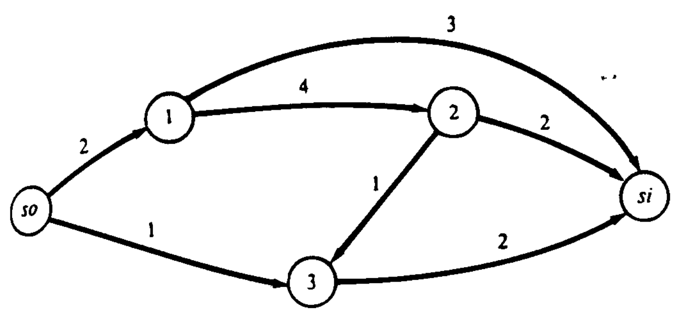


FIGURE 20 Network for Problem 3

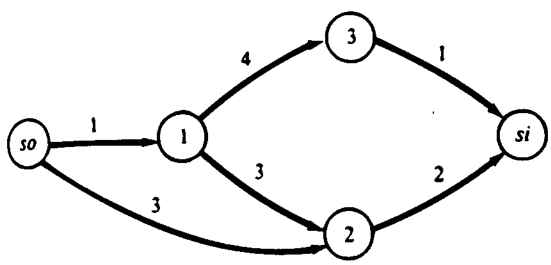


FIGURE 21 Network for Problem 4

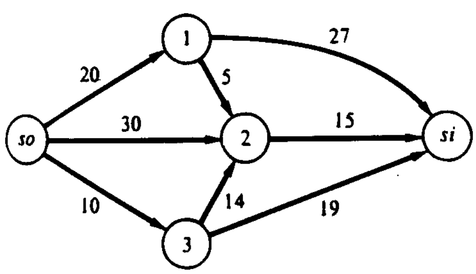
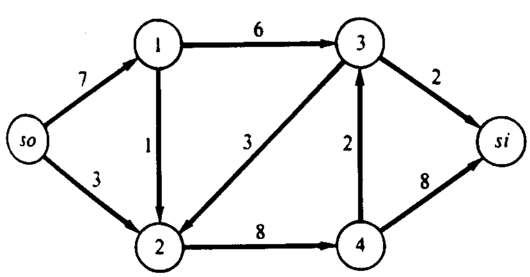


FIGURE 22 Network for Problem 5



used to determine whether the packages can be loaded so that no truck carries two packages of the same type.

7 Four workers are available to perform jobs 1–4. Unfortunately, three workers can do only certain jobs: worker 1, only job 1; worker 2, only jobs 1 and 2; worker 3, only job 2; worker 4, any job. Draw the network for the maximum flow problem that can be used to determine whether all jobs can be assigned to a suitable worker.

8 The Hatfields, Montagues, McCoys, and Capulets are going on their annual family picnic. Four cars are available to transport the families to the picnic. The cars can carry the following number of people: car 1, four; car 2, three; car 3, three; and car 4, four. There are four people in each family, and no car can carry more than two people from any one family. Formulate the problem of transporting the maximum possible number of people to the picnic as a maximum flow problem.

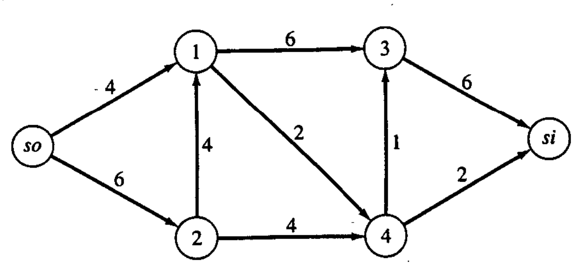
9–10 For the networks in Figures 23 and 24, find the maximum flow from source to sink. Also find a cut whose capacity equals the maximum flow in the network.

Group B

11 Suppose a network contains a finite number of arcs and the capacity of each arc is an integer. Explain why the Ford–Fulkerson method will find the maximum flow in the finite number of steps. Also show that the maximum flow from source to sink will be an integer.

12 Consider a network flow problem with several sources and several sinks in which the goal is to maximize the total

FIGURE 23



Assignment Problem

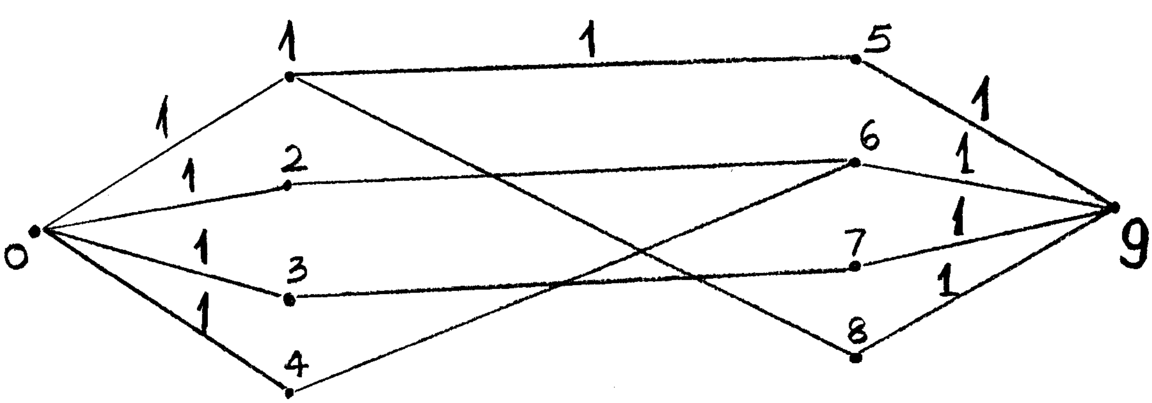
5	7	9	6
11	4	13	17
8	9	4	13
10	7	11	9

$\Rightarrow$

0	2	4	1
7	0	9	13
4	5	0	9
3	0	4	2

$\Downarrow$

0^*	2	4	0
7	0^*	9	12
4	5	0^*	8
3	0	4	1

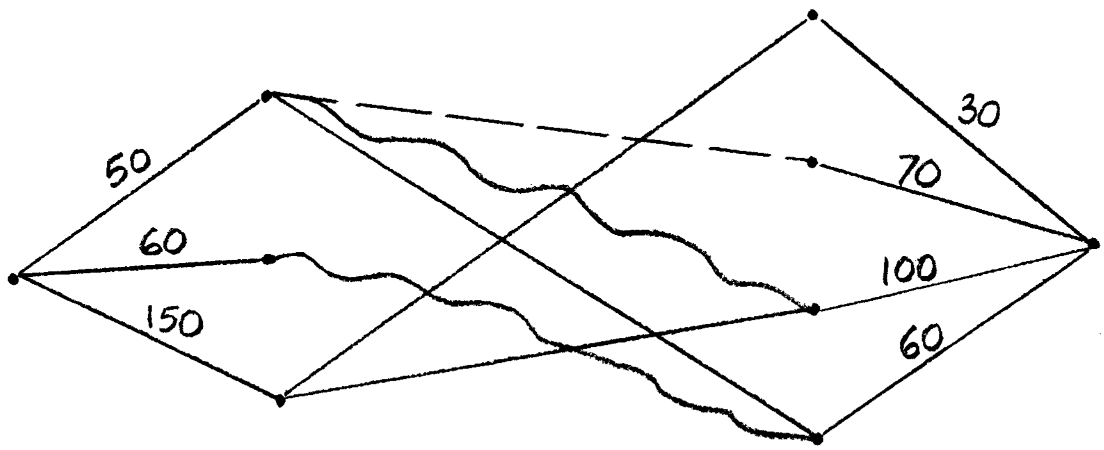


8	6	10	5	40
11	4	3	8	60
9	6	7	11	50
25	35	50	40	

Transportation Problem

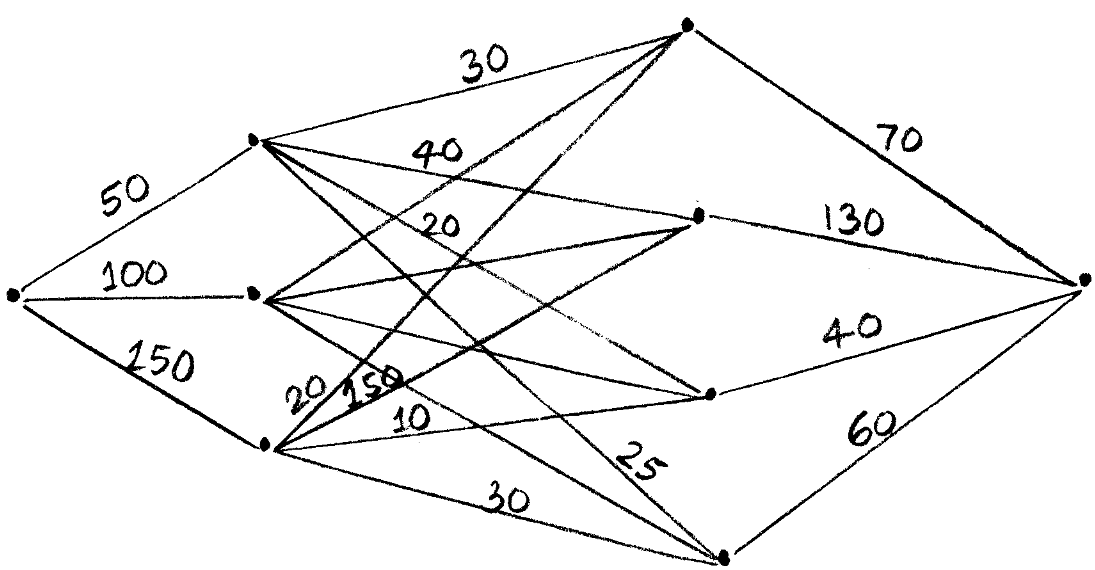
Time criteria

5	2	4	1	50
8	12	6	4	60
1	5	1	8	150
30	70	100	60	



Transportation with bounded capacities

3/30	14/40	15/20	6/25	50
8/35	10/20	12/15	9/40	100
2/20	8/100	7/10	17/30	150
70	130	40	60	



Transportation with bounded capacities

3 / 30	14 / 50	15 / 20	6 / 25	50	−3
8 / 40	10 / 20	12 / 40	9 / 30	100	−8
6 / 20	8 / 70	7 / 80	11 / 100	150	−6
70	130	40	60		

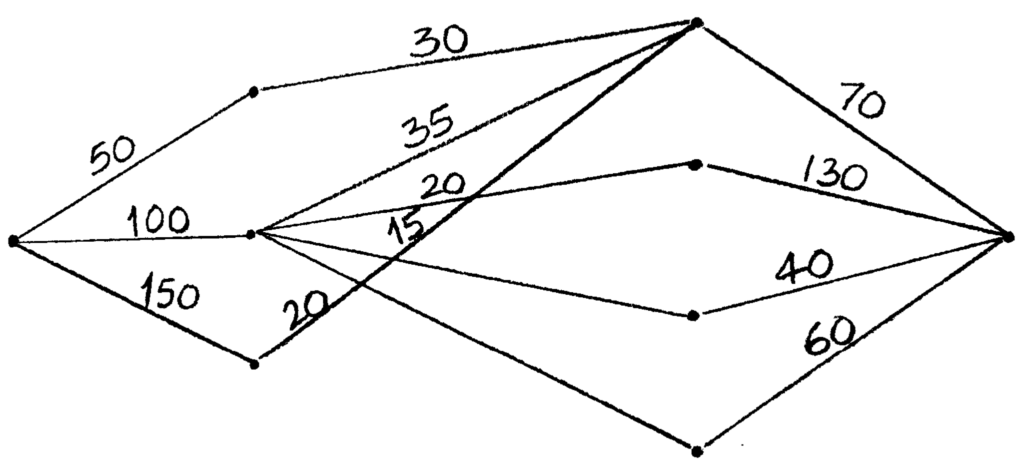
⇓

0	11	12	3
0	2	4	1
0	2	1	8
	−2	−1	−1

0	11	12	3	50
0	2	4	1	100
0	6	5	12	150
70	130	40	60	

⇓

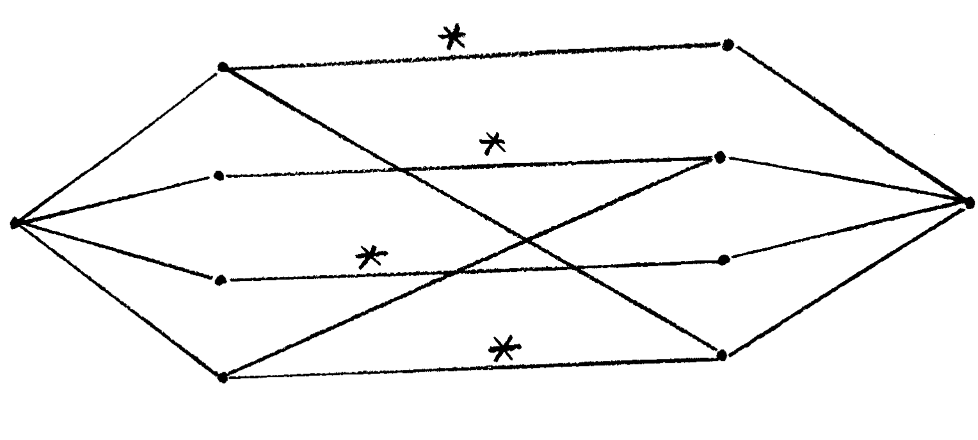
0 / 30	9 / 40	8 / 20	2 / 25	50
0 / 35	0 / 20	0 / 15	0 / 40	100
0 / 20	4 / 100	1 / 10	11 / 30	150
70	130	40	60	



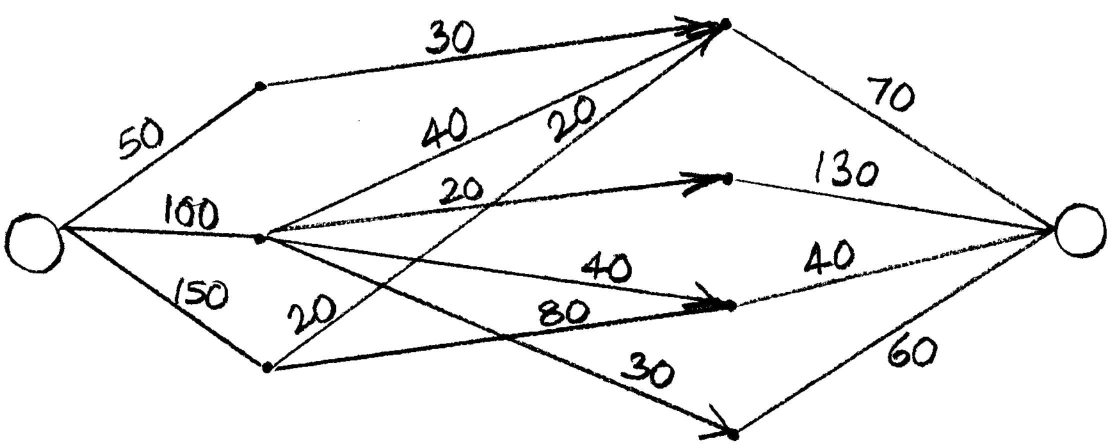
$\mu(0, 1, 5, 9)$, $\mu(0, 2, 6, 9)$, $\mu(0, 3, 7, 9)$

0	2	4	0	
7	0	9	12	−1
4	5	0	8	
3	0	4	1	−1
	+1			

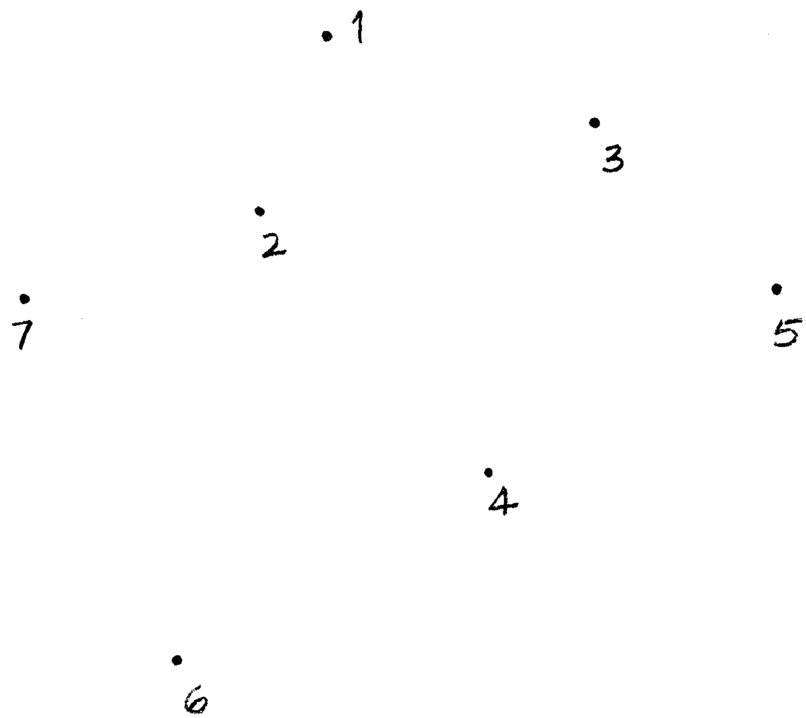
0	3	4	0
6	0	8	11
4	6	0	8
3	0	3	0

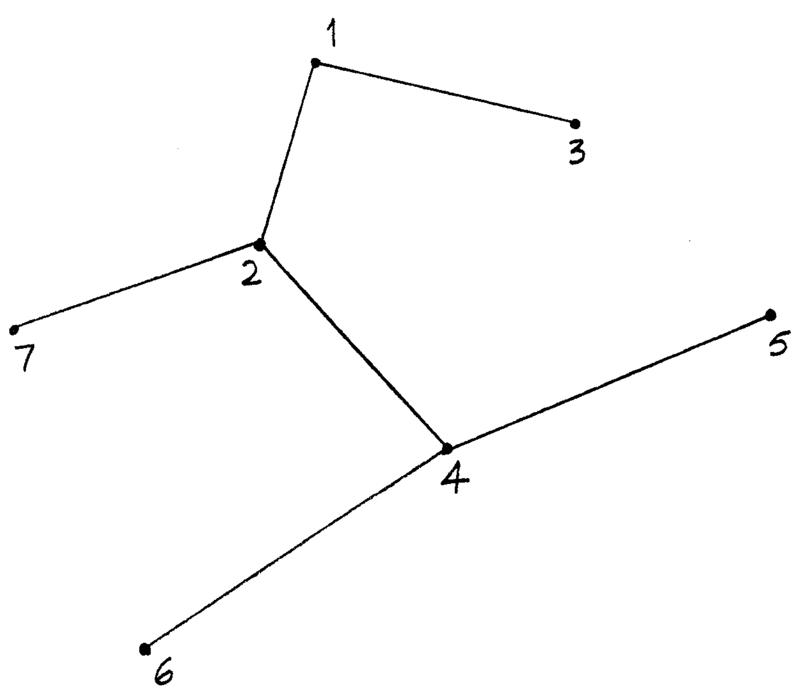


0	9	11	2
0	0	3	0
0	0	0	7



Minimum Spanning Tree Problem





Transportation Min – Time

3	14	15	6	50
8	10	12	9	100
2	8	7	14	150
70	130	40	60	